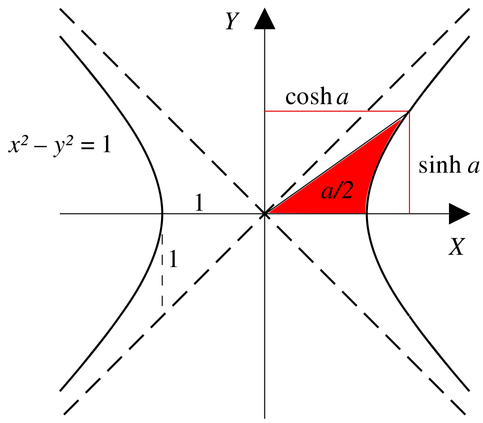
Math 135, Calculus 1, Fall 2020

Project 2: Hyperbolic Functions.
Due: **Friday, October 30** by 11:59pm.

The Project

In this project, you will investigate a new family of functions called **hyperbolic functions**. These are analogues of the ordinary trigonometric functions based on the "unit byperbola $x^2 - y^2 = 1$ as opposed to the unit circle:



Components. This project will consist of a single document, your report. This will include the answer to a variety of questions, as well as several graphs and visual analyses.

Setup. There are explicit formula, in terms of the exponential function e^x, for the basic hyperbolic functions:

- **hyperbolic sine** is defined to be
$$\sinh t = \frac{e^t - e^{-t}}{2}.$$
- **hyperbolic cosine** is defined to be
$$\cosh t = \frac{e^t + e^{-t}}{2}.$$
- **hyperbolic tangent** is defined to be
$$\tanh t = \frac{\sinh t}{\cosh t}.$$
- **hyperbolic secant** is defined to be
$$\operatorname{sech} t = \frac{1}{\cosh t}.$$

A. Graphs of hyperbolic functions

We begin by investigating the graphs of these functions.

Problem 1.

(a) Graph the four hyperbolic functions above, and include these in your report.
(b) Based on these graphs, what is the domain and range of each of these functions?
(c) Confirm your answer for the **domains** using algebra. How does this relate to the analogous trig functions?

Problem 2.

(a) Based on these graphs, what are the horizontal asymptotes for $\tanh t$? Confirm your answer using algebra.
(b) Graph $\tanh t$ and $\arctan t$ on the same set of axes, and include this in your report. What similarlities do you notice?

Problem 3.

(a) Based on these graphs, for each of these four hyperbolic functions, decide if the function is: even, odd, or neither.
(b) Using algebra and the definition of even and odd functions, prove your assertions from Part 3a.

Problem 4.

(a) On `desmos.com/calculator`, draw $(\cosh t, \sinh t)$ for $-2 \leq t \leq 2$. Include this in your report.
(b) Using the explicit definitions from the introduction, show the identity $\cosh^2 t - \sinh^2 t = 1$, confirming that $(\cosh(t), \sinh(t))$ lie on the unit hyperbola $x^2 - y^2 = 1$.

B. Calculus of hyperbolic functions

The derivative formulas for the hyperbolic functions are very similar (but not identical) to those for the trigonometric functions:

Problem 5. Show that $\frac{d}{dt}\Big(\sinh t\Big) = \cosh t$.

Problem 6.

(a) Find formula for $\frac{d}{dt}\Big(\cosh t\Big)$ and $\frac{d}{dt}\Big(\tanh t\Big)$ in terms of other hyperbolic functions. Show all work that led to these formulas.
(b) What simple second-order differential equation is $y = \sinh t$ a solution?
(*Hint: see Written HW 10-23, Exercise 1.*)

C. Inverses of hyperbolic functions

We would like to compute the inverse functions for some of these hyperbolic functions.

Problem 7. Use the explicit definition from the introduction to find the exact value of t such that $\sinh t = \dfrac{3}{4}$.
(*Hint: Leave your answer as a logarithm.*)

Problem 8.

(a) Graphically, it is clear that $\sinh t$ passes the horizontal line test, and hence is 1-to-1. Confirm this by using calculus to show that $\sinh t$ is always increasing.

(b) Show that $\sinh^{-1}(y) = \ln(y + \sqrt{y^2 + 1})$.
(*Hint: When solving for t in $\sinh t = y$, first set $e^t = z$ and solve for z.*
*You will need to use the quadratic formula. Why is only **one** of the possible solutions valid?*)

Problem 9. We will find the derivative of $\sinh^{-1}(t)$ in two different ways:

(a) Use Implicit Differentiation, as used in Worksheet 10-26 to compute $\dfrac{d}{dx}(\cos^{-1}(x))$, to compute the derivative of $\sinh^{-1}(t)$.

(b) Use the formula $\sinh^{-1}(t) = \ln\left(t + \sqrt{t^2 + 1}\right)$ and $\dfrac{d}{dt}(\ln(t)) = \frac{1}{t}$ (see Worksheet for 10-26) to compute the derivative of $\sinh^{-1}(t)$.