Math 135, Calculus 1, Fall 2020

Project 3: Approximating Functions
Due: **Friday, Dec 11** by 11:59pm.

The Project

In this project, you will investigate the function $f(x) = e^x \sin(x)$ on the domain $[-2\pi, 2\pi]$, as well as several approximations of this function.

Components. For this project, you will need to submit mathematical solutions to various questions, as well as a single graph. You must submit your project as a single file or Google Document.

For these questions, you must **show all work**, and use the Calculus techniques discussed in class. Answers not supported in this way will **not** be accepted.

A. The function $f(x) = e^x \sin(x)$ [40 points]

Problem 1. Find all critical points of $f(x)$ on $[-2\pi, 2\pi]$. How many are there?

Problem 2. Create a sign chart for the first derivative of $f(x)$ on $[-2\pi, 2\pi]$. Show all work.

Problem 3. Create a sign chart for the second derivative of $f(x)$ on $[-2\pi, 2\pi]$. Show all work. Find the inflection points on $[-2\pi, 2\pi]$. How many are there?

Problem 4. Classify the critical points of $f(x)$ on $[-2\pi, 2\pi]$ using both the First and Second Derivative Tests.

Problem 5. Find the absolute maximum and absolute minimum values of $f(x)$ on $[-2\pi, 2\pi]$ using the Extreme Value Theorem.

B. Approximations of $f(x) = e^x \sin(x)$ [45 points]

B.1. **Linearization.** The *linearization* of a function $f(x)$ at $x = a$ is a linear function

$$L(x) = A + Bx$$

such that

$$L(a) = f(a), \qquad L'(a) = f'(a).$$

$L(x)$ gives the best approximation of $f(x)$ near $x = a$ as a line.

Problem 6. Find $L(x)$ for $f(x) = e^x \sin(x)$ at $x = 0$. Show all work.
(Hint: We know that $L(x)$ must be of the form $A + Bx$. Can we use the properties of $L(x)$ listed above to find the values of A and B?)

Problem 7. How many critical points does $L(x)$ have in $[-2\pi, 2\pi]$? How many inflection points does $L(x)$ have in $[-2\pi, 2\pi]$?

Problem 8. Compute the absolute max and min values of $L(x)$ on $[-2\pi, 2\pi]$ using the EVT.

B.2. **Quadratic approximation.** The *quadratic approximation* of a function $f(x)$ at $x = a$ is a quadratic function

$$Q(x) = A + Bx + Cx^2$$

such that

$$Q(a) = f(a), \qquad Q'(a) = f'(a), \qquad Q''(a) = f''(a).$$

$Q(x)$ gives the best approximation of $f(x)$ near $x = a$ as a parabola.

Problem 9. Find $Q(x)$ for $f(x) = e^x \sin(x)$ at $x = 0$. Show all work.
(Hint: Can we use the properties of $Q(x)$ listed above to find the necessary values of A, B, and C?)

Problem 10. How many critical points does $Q(x)$ have in $[-2\pi, 2\pi]$? How many inflection points does $Q(x)$ have in $[-2\pi, 2\pi]$?

Problem 11. Compute the absolute max and min values of $Q(x)$ on $[-2\pi, 2\pi]$ using the EVT.

B.3. **Cubic approximation.** The *cubic approximation* of a function $f(x)$ at $x = a$ is a cubic function

$$P(x) = A + Bx + Cx^2 + Dx^3$$

such that

$$P(a) = f(a), \qquad P'(a) = f'(a), \qquad P''(a) = f''(a), \qquad P'''(a) = f'''(a).$$

$P(x)$ is the best approximation of $f(x)$ near $x = a$ as a cubic function.

Problem 12. Find $P(x)$ for $f(x) = e^x \sin(x)$ at $x = 0$. Show all work.
(Hint: Can we use the properties of $O(x)$ listed above to find the necessary values of A, B, C, and D?)

Problem 13. How many critical points does $P(x)$ have in $[-2\pi, 2\pi]$? How many inflection points does $P(x)$ have in $[-2\pi, 2\pi]$?

Problem 14. Compute the absolute max and min values of $P(x)$ on $[-2\pi, 2\pi]$ using the EVT.

B.4. **Graph.**

Problem 15. Using Desmos, plot $f(x)$, $L(x)$, $Q(x)$, and $P(x)$ on the same set of axes. Make sure the domain is at least $[-2\pi, 2\pi]$.

C. General Approximations [15 points]

Suppose $f(x)$ is a function such that $f(x)$, $f'(x)$, $f''(x)$, and $f'''(x)$ are all continuous, and let $x = a$ be some point in the domain of f.

The following formula should be in terms of a, $f(a)$, $f'(a)$, $f''(a)$, and $f'''(a)$.

Problem 16. Find a general rule for $L(x)$.

Problem 17. Find a general rule for $Q(x)$.

Problem 18. Find a general rule for $P(x)$.