Math 135, Calculus 1, Fall 2020

10-14: Product of Quotient Rules

Last week, we introduced the **derivative function** $f'(x)$ of a function $f(x)$, whose evaluation $f'(a)$ at the point $x = a$ is give by:

- the slope of the tangent line at $x = a$
- the instantaneous velocity at time $x = a$

On Monday, we covered algorithms to help us compute the derivatives of polynomials and exponential functions. Today, we'll tackle **products** and **quotients**

Theorem (Product Rule). *If $f(x)$ and $g(x)$ are differentiable functions, then so is their product $f(x) \cdot g(x)$. The derivative of the product is given by*

$$\frac{d}{dx}\Big(f(x) \cdot g(x)\Big) = f'(x) \cdot g(x) + f(x) \cdot g'(x).$$

Example 1 (Warning). The derivative of a product is **not** equal to the product of the derivatives:

Consider the product $x \cdot x$. Then

$$\frac{d}{dx}\big(x\big) \cdot \frac{d}{dx}\big(x\big) = 1 \cdot 1 = 1$$

but instead

$$\frac{d}{dx}\big(x \cdot x\big) = \frac{d}{dx}\big(x\big) \cdot x + x \cdot \frac{d}{dx}\big(x\big) = 1 \cdot x + x \cdot 1 = 2x$$

which is what the **power rule** gives us for the derivative of $x \cdot x = x^2$.

Exercise 1. Use the Product Rule to find $f'(x)$ when $f(x) = (3x^2 + 1)e^x$. Simlify your answer.

Theorem (Quotient Rule). *If $f(x)$ and $g(x)$ are differentiable functions, then so is their quotient $f(x)/g(x)$ whenever $g(x) \neq 0$. The derivative of the quotient is given by*

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}.$$

Remark 2. The Quotient Rule can be derived from the product rule: Letting $Q(x) = \dfrac{f(x)}{g(x)}$, cross multiply the equation and differentiate both sides with respect to x using the Product Rule. Solving for $Q'(x)$ leads to the above formula.

Exercise 2. Use the Quotient Rule to calculate the derivative of $\frac{1}{x^4}$ and check your answer against the result obtained from using the Power Rule.

Exercise 3. If $g(x) = \frac{3x+1}{2x-5}$, find a simplify $g'(x)$.

Exercise 4. If $h(x) = \frac{e^x}{x^2+1}$, find and simplify $h'(x)$.

Exercise 5. Suppose that $f(3) = 5$, $f'(3) = -7$, $g(3) = 2$, and $g'(3) = 1/2$. If $H(x) = \frac{f(x)}{xg(x)}$, find $H'(3)$.