Math 135, Calculus 1, Fall 2020

10-16: Higher Order Derivatives (Section 3.5) and Trig Derivatives (Section 3.6)

Last week, we introduced the **derivative function** $f'(x)$ of a function $f(x)$, whose evaluation $f'(a)$ at the point $x = a$ is give by:

- the slope of the tangent line at $x = a$
- the instantaneous velocity at time $x = a$
- the instantaneous rate of change of f with respect to x

Today, we'll be discussing **higher order derivatives** and the derivatives of **trig functions**.

A. Higher Derivatives

The derivative $f'(x)$ of a function $f(x)$ gives the **slope** of the function f at the point x. However, $f'(x)$ is also a function, and so we can ask: "What is the derivative of the derivative?"

Definition 1. The **second derivative** of $f(x)$ is the function

$$f''(x) = \frac{d}{dx}\big(f'(x)\big) = \frac{d^2 f}{dx^2} = \frac{d^2 y}{dx^2}.$$

To compute the second derivative, we simply take the derivative twice.

Exercise 1. Suppose that $f(x) = 2x^4 - 3x + e^x$. Find $f'(x)$ and $f''(x)$.

A.1. The second derivative and concavity.

The second derivative measures the change in $f'(x)$, i.e. the change in the slope of $f(x)$. What does this really mean?

Recall:

- $\frac{d}{dx}(f)|_{x=a} > 0 \quad \Leftrightarrow \quad$ the slope is positive at $x = a \quad \Leftrightarrow \quad f(x)$ is **increasing** at $x = a$
- $\frac{d}{dx}(f)|_{x=a} < 0 \quad \Leftrightarrow \quad$ the slope is negative at $x = a \quad \Leftrightarrow \quad f(x)$ is **decreasing** at $x = a$

If $f''(a) > 0$, then $\frac{d}{dx}(f')|_{x=a} > 0$, so the slopes of f are increasing at $x = a$. There are two options:

- If the slopes are already positive, then they are getting bigger, so the curve is getting steeper, increasing at a faster rate (like e^x)
- If the slopes are negative, then the function $f(x)$ is still decreasing, but beginning to flatten out: the negative slopes are increasing towards (and possibly past) zero.

In these cases, we say the graph of f is **concave up** at $x = a$.

Similarly, the reverse options can happen if $f''(a) < 0$, and the graph is **concave down**.

Exercise 2. Sketch the graph of a function g such that $g'(x) < 0$ and $g''(x) > 0$ everywhere.

Exercise 3. Former President Nixon famously said, "Although the rate of inflation is increasing, it is increasing at a decreasing rate." Let $r(t)$ denote the rate of inflation. According to President Nixon, what are the signs (+, −, or 0) of $r'(t)$ and $r''(t)$?

A.2. **Higher Derivatives.** .

In good cases, we can continue to take derivatives of derivatives.

- We write $f'''(x)$ for the **third derivative** $\frac{d}{dx}(f''(x))$.
- More generally, we write $f^{(n)}(x)$ for the n**-th derivative** of $f(x)$.

Exercise 4. Suppose that $f(x) = xe^x$.

(a) Find and simplify $f'(x)$, $f''(x)$, and $f'''(x)$.

(b) Find a general formula, in terms of n, of the n-th derivative $f^{(n)}(x)$.

Exercise 5. The inflation rate is given by the (positive) rate of change of the **consumer price index**. Let $p(t)$ denote the consumer price index. According to Nixon, what are the signs (+, −, or 0) of $p'(t)$, $p''(t)$, and $p'''(t)$?

Exercise 6. A news report out of Massachusetts yesterday said:

> The total number of COVID cases that were confirmed last week grew to 4,560 today. That's a 12% increase over the previous week and an 83% increase in cases over the week of Sept. 13, when cases began to rise at a higher rate.

Let $N(t)$ denote the cumulative total number of cases of COVID-19 in Massachusetts. What derivative of N went from negative to positive on September 13? Using evidence from the news article, is that derivative still positive?

B. Trig Derivatives

Using the trig identities $\sin(A+B) = \sin A \cdot \cos B + \cos A \sin B$, along with the trig limits $\lim_{x\to 0} \frac{\sin x}{x} = 1$ and $\lim_{x\to 0} \frac{1-\cos x}{x} = 0$, we can compute the derivatives of $\sin(x)$ and $\cos(x)$:

Theorem 2. *If x is measured in radians, then*

$$\frac{d}{dx}(\sin x) = \cos x, \qquad \frac{d}{dx}(\cos x) = -\sin x.$$

To prove the first formula, let $f(x) = \sin x$. Then

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{\sin(x+h) - \sin(x)}{h} \\ &= \lim_{h\to 0} \frac{\sin x \cdot \cos h + \cos x \cdot \sin h - \sin x}{h} \\ &= \lim_{h\to 0} \frac{\sin x \cdot (\cos h - 1) + \cos x \cdot \sin h}{h} \\ &= \lim_{h\to 0} \sin x \cdot \frac{\cos h - 1}{h} \quad + \quad \lim_{h\to 0} \cos x \cdot \frac{\sin h}{h} \\ &= \sin x \cdot 0 + \cos x \cdot 1 \\ &= \cos x. \end{aligned}$$

Exercise 7. Use the quotient rule and the above results to prove that $\frac{d}{dx}(\tan x) = \sec^2 x$.

Exercise 8. Show that $\frac{d}{dx}(\sec x) = \sec x \cdot \tan x$.

Exercise 9. If $g(x) = x^3 \sin x$, find a simplify $g'(x)$ and $g''(x)$.