# Math 135, Calculus 1, Fall 2020

10-19: The Chain Rule (Section 3.7)

Last week, we introduced the **derivative function** $f'(x)$ of a function $f(x)$, whose evaluation $f'(a)$ at the point $x = a$ is give by:

- the slope of the tangent line at $x = a$
- the instantaneous velocity at time $x = a$
- the instantaneous rate of change of $f$ with respect to $x$

Today: Trig derivatives and the chain rule.

## A. Trig Derivatives

**Theorem 1.** *If $x$ is measured in radians, then*

$$\frac{d}{dx}(\sin x) = \cos x, \qquad \frac{d}{dx}(\cos x) = -\sin x.$$

The proof can be found in Worksheet 10-16.

**Exercise 1.** Use the quotient rule and the above results to prove that $\frac{d}{dx}(\cot x) = -\csc^2 x$.

## B. Chain Rule

The chain rule gives us a way to compute the derivative of a **composite** of two functions.

**Theorem 2.** *If $O(x)$ and $I(x)$ are differentiable functions, then so is the composite $O(I(x)) = (O \circ I)(x)$. Moreover,*

$$\frac{d}{dx}\Big(O(I(x))\Big) = O'\big(I(x)\big) \cdot I'(x).$$

That is, "the derivative of the outside function **evaluated at the inside function**, times the derivative of the inside function".

**Example 3.** Let $h(x) = (x^4+1)^2$. The **inside function** is $I(x) = x^4 + 1$, as this is the **first** thing we do to evaluate this function. The **outside function** is $O(x) = x^2$, as this is the **second** thing we do to evaluate this function. Then $h(x) = O(I(x))$.

Since $O'(x) = 2x$ and $I'(x) = 4x^3$, we have $O'(I(x)) = 2 \cdot (x^4+1)$, and the Chain Rule says that

$$h'(x) = O'(I(x)) \cdot I'(x) = 2(x^4+1) \cdot 4x^3 = 8x^7 + 8x^3.$$

To check this, we can first exapnd $h(x)$ and compute $h'(x)$ via the Power Rule:

$$h(x) = (x^4+1)^2 = x^8 + 2x^4 + 1, \qquad h'(x) = 8x^7 + 8x^3.$$

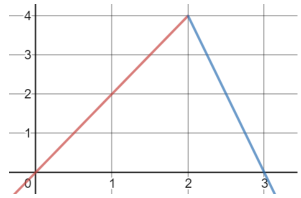
**Exercise 2.** Let $f(x) = \sin(2x)$.

(a) Use the double-angle formula $\sin(2x) = 2\sin x \cos x$ to compute $f'(x)$ using the product rule.

(b) Use the Chain Rule to directly compute $f'(x)$. (*What is the inside function? Outside function?*)

**Exercise 3.** Let $f(x)$ and $g(x)$ be two functions. Certain values of $f(x)$ and $f'(x)$ are given in the table below, and the graph of $g(x)$ is as shown.

| x | 1 | 2 | 3 |
|---|---|---|---|
| f(x) | 3 | 2 | 1 |
| f′(x) | 4 | 5 | 6 |




(a) Let $h(x) = g(f(x))$. Find $h'(3)$.

*Computation.* The Chain Rule says that $h'(x) = g'(f(x)) \cdot f'(x)$, so

$$h'(3) = g'(f(3)) \cdot f'(3) = g'(1) \cdot 6 = 2 \cdot 6 = 12.$$

□

(b) Let $k(x) = f(g(x))$. Find $k'(1)$.

**Exercise 4.** If $F(x) = \sqrt{x^4 + 3}$, use the Chain Rule to find and simplify $F'(x)$.

**Exercise 5.** (a) If $y = e^{x^2}$, compute $\frac{dy}{dx}$.

(b) If $z = e^{\tan t}$, compute $\frac{dz}{dt}$.

You may need to use the Chain Rule multiple times:

**Exercise 6.** Find and simplify $G'(x)$ if $G(x) = \sin\left(\sqrt{x^2+2}\right) + e^{\cos(4x)}$.