Math 135, Calculus 1, Fall 2020

11-30: L'Hôpital's Rule

The **derivative** $f'(x)$ of a function $y = f(x)$ gives:

- the slope of the tangent line
- the instantaneous rate of change of y with respect to x

Common Problem. Suppose we want to compute the limit of a rational function

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

but when we plug in $x = a$ (including $a = \pm\infty$), we get one of the following **indeterminant forms**:

$$\frac{0}{0} \qquad \text{or} \qquad \frac{\pm\infty}{\infty}$$

In this case, we can apply L'Hôpital's Rule to help compute this limit.

Theorem 1 (L'Hôpital's Rule, Guillaume François Antoine Marquis de L'Hôpital, 1696)**.** *If f and g are differentiable functions such that $\frac{f(a)}{g(a)}$ is one of the above indeterminant forms, then*

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}.$$

- You may have to apply the rule multiple times to determine the limit
- The rule was actually discovered by Bernoulli in 1964.

Example 2. Consider $\lim_{x \to 0} \frac{\sin x}{x}$, where x is in radians. Plugging in $x = 0$ yields the indeterminant form $\frac{0}{0}$, so L'Hôpital's Rule applies. Thus we have

$$\lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} \frac{\cos x}{1} = \frac{\cos 0}{1} = 1.$$

This matches our calculation from Section 2.6.

Exercise 1. Use L'Hôpital's Rule to compute the other important trig limit

$$\lim_{x \to 0} \frac{1 - \cos x}{x} =$$

Exercise 2. Ccompute the following limits. Make sure to **first check** that the rule actually applies.

(a) $\lim_{x \to 2} \frac{x^3 - 8}{x^4 - x^3 - 8}$

(b) $\lim_{x \to 0} \frac{1 - \cos x}{x^2}$

(c) $\lim_{x \to 3} \frac{x - 3}{x}$

(d) $\lim_{x \to \infty} \frac{\ln x}{\sqrt{x}}$

(e) $\lim_{x \to \infty} \frac{e^x}{x}$

Exercise 3. Use L'Hôpital's Rule to find any horizontal asymptotes of the following functions.

(a) $f(x) = \frac{8x^3 - 4x + \pi}{-3x^3 + 7x^2 + 5}$

(b) $g(x) = \frac{3e^{2x} + 5x}{e^{2x} + 8x}$