Math 135, Calculus 1, Fall 2020

09-11: Composition and Inverses (Section 1.5)

A. Meet your classmates

Share and discuss your experience (past or desired) at Holy Cross:

Question 1. *What classes are you taking this year, or want to take in the future? What clubs are you excited to check out, or have been part of? Why did you choose Holy Cross?*

B. Composition

Given two functions f and g, we may consider them in succession by **composing them together**. This creates a new function:

> The **composite** of f with g is the new function $f \circ g$ given by
> $$(f \circ g)(x) = f(g(x)) = \text{"}f \text{ of } g \text{ of } x\text{"}$$

Exercise 2. Suppose that $f(x) = \sqrt{x}$ and $g(x) = 3x + 1$. Find $(f \circ g)(x)$ and $g(f(x))$.

$(f \circ g)(x) = \sqrt{3x+1}$

$g(f(x)) = 3\sqrt{x} + 1$

What is the domain of $(f \circ g)(x)$? How does it compare to the domain of g?

$3x+1 \geq 0$, or

$\boxed{x \geq -1/3}$ so smaller!

$\overline{(-\infty, \infty)}$

What is the domain of $g(f(x))$? How does it compare to the domain of f?

$\boxed{x \geq 0}$ so same!

$\overline{x \geq 0}$

Exercise 3. Suppose that $f(x) = 5x - 3$. Find a function $h(x)$ such that $h(f(x)) = x$. (Hint: switch the role of y and x in the rule for $f(x)$)

$y = 5x - 3 \leadsto x = 5y - 3$

$\boxed{h(x) = y = \frac{1}{5}x + \frac{3}{5}}$

$h(f(x)) = \frac{1}{5}(5x-3) + 3/5 = x - \frac{3}{5} + 3/5 = x$ ✓

C. Inverses

We may try to "reverse" a function: if $f(2) = 7$, is there a new function f^{-1} such that $f^{-1}(7) = 2$?

Exercise 4. Consider $f(x) = 2x - 8$ and $g(x) = \frac{1}{2}x + 4$.

(a) We see $f(0) = -8$. What is $g(-8)$? 0

(b) We see $f(3) = -2$. What is $g(-2)$? -2

(c) What is $f(g(x))$?

$$2\left(\tfrac{1}{2}x + 4\right) - 8 = x + 8 - 8 = \boxed{x}$$

(d) What is $g(f(x))$?

$$\tfrac{1}{2}(2x-8)+4 = x - 4 + 4 = \boxed{x}$$

The **inverse** of a function f, if it exists, is another function f^{-1} such that

$$f(f^{-1}(x)) = x \qquad \text{and} \qquad f^{-1}(f(x)) = x.$$

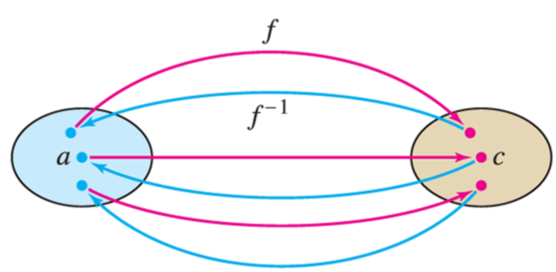
Warning: $f^{-1}(x)$ is NOT the same as $\dfrac{1}{f(x)}$.

Definition C.1. A function f is **one-to-one** on a domain D if, for every value c, the equation $f(x) = c$ has *at most* one solution x in D.

This is true exactly when the graph of the function passes the **horizontal line test**.

Theorem C.2. *The inverse function $f^{-1}(x)$ exists exactly when f is one-to-one on its domain.*

Domain of f = range of f^{-1}



Range of f = domain of f^{-1}

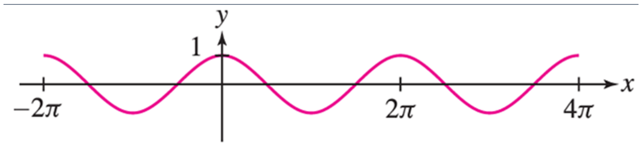
Exercise 5. Why doesn't $f(x) = x^2$ have an inverse? Can we shrink the domain of f so that it does?

- Isn't 1-to-1: e.g. $f(2) = f(-2) = 4$
- Shrink domain to $[0, \infty)$: $f^{-1}(x) = \sqrt{x}$

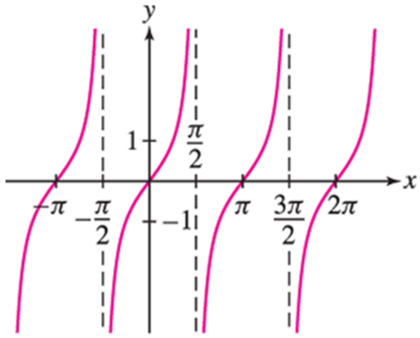
$f^{-1}(4)$ = "the unique x in $[0,$

· Shrink domain to $(-\infty, 0]$: $f^{-1}(x) = -\sqrt{x}$

D. Inverse Trig Functions



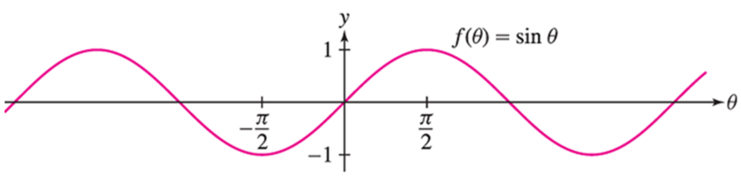
$y = \cos x$

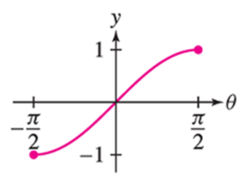


$y = \tan x$

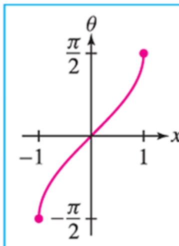
None of the trig functions satisfy the horizontal line test. This means that they don't have inverses over their whole domain. Instead, we **restrict** the domain of each trig function so that they do.

For example, to define the inverse of $\sin\theta$, denoted $\sin^{-1}(\theta)$ or $\arcsin\theta$, we restrict the domain of $\sin\theta$ to $[-\pi/2, \pi/2]$, where it passes the horizontal line test.





sin θ with restricted domain



$\theta = \sin^{-1} x$

Exercise 6. We have stated the restricted domain we will use for each trig function below. Complete the rest of the table.

function	restricted domain	range	inverse function	domain	range
$\cos(x)$	$[0, \pi]$		$\cos^{-1}(x)$		
$\sin(x)$	$[-\pi/2, \pi/2]$		$\sin^{-1}(x)$		
$\tan(x)$	$(-\pi/2, \pi/2)$		$\tan^{-1}(x)$		

Exercise 7. Find all solutions $0 \le \theta \le 2\pi$ to $\sin\theta = \dfrac{-1}{2}$. How does your answer compare to $\sin^{-1}\left(\dfrac{-1}{2}\right)$?