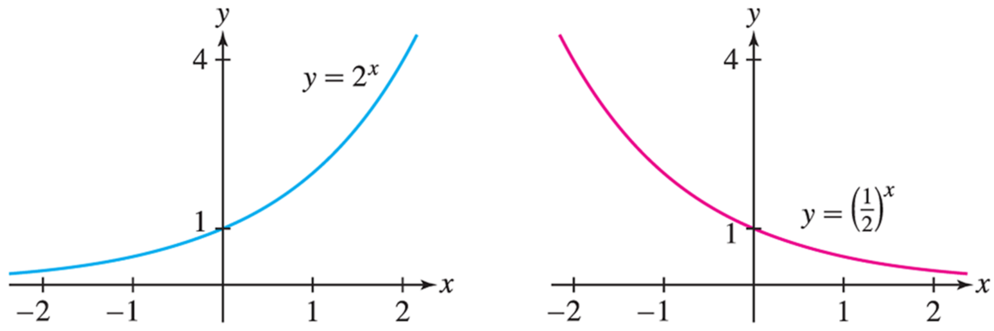
Math 135, Calculus 1, Fall 2020

09-14: Exponentials and the Logarithm (Section 1.6)

A. Exponential Functions and the Logarithm

Exponential functions are of the form $f(x) = b^x$ with $b > 0$. If $b > 1$, $f(x)$ is *increasing*, while if $0 < b < 1$, $f(x)$ is *decreasing*.



$y = 2^x$ is increasing $\qquad$ $y = \left(\frac{1}{2}\right)^x$ is decreasing

Exponential functions are 1-1, so $b^x = b^t$ exactly when $x = t$.

Exercise 1. Suppose $3^{x+1} = \left(\frac{1}{3}\right)^{2x}$. Solve for x.

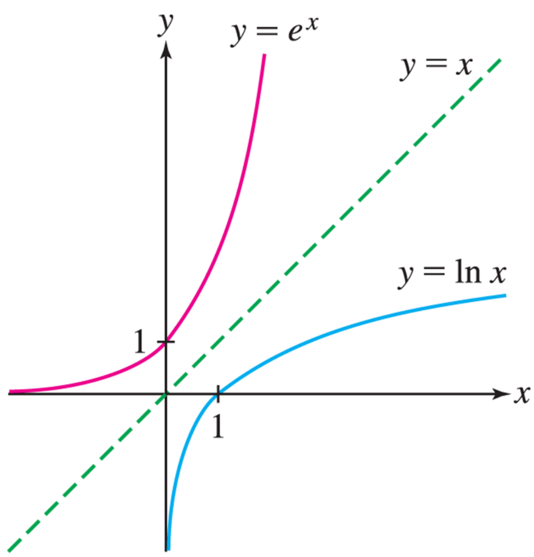
$\left(\frac{1}{3}\right)^{2x} = (3^{-1})^{2x} = 3^{-2x}$

$3^{x+1} = 3^{-2x} \iff x+1 = -2x$

$3x = -1$

$\boxed{x = -1/3}$

The **inverse** of the exponential function $f(x) = b^x$ is called the **logarithm with base** b, denoted $\log_b(x)$.



Using the definition of the inverse, we have:

$$y = b^x \quad \text{exactly when} \quad \log_b(y) = x$$

Exercise 2. Use the above to compute $b^{\log_b x}$ and $\log_b(b^x)$.
(Hint: **inverses!** The answer should not be complicated.)

- $y = b^{(\log_b x)} \iff \log_b(y) = \log_b(x) \iff y = \boxed{x}$
- $\log_b(b^x) = y \iff b^x = b^y \iff y = \boxed{x}$

Exercise 3. Compute $\log_3(27)$. (Hint: let $x = \log_3(27)$. Now apply the above framed box.)

$x = \log_3(27) \iff 27 = 3^x \iff \boxed{x = 3}$

Euler's number is the irrational number $e \approx 2.718$. The associated exponential e^x has good properties (to be discovered later). The associated logarithm $\log_e(x) = \ln(x)$ is called the **natural logarithm**.

Exercise 4. Compute the domain and range of $\ln(x)$.

function	domain	range
e^x	$(-\infty, \infty)$	$(0, \infty)$
$\ln(x)$	$(0, \infty)$	$(-\infty, \infty)$

Exercise 5. Use the laws of exponents to compute the following without a calculator; all answers are integers. (Hint: use the framed box on the previous page.)

(a) $\log_b(1)$

$\log_b(1) = x \iff 1 = b^x \iff \boxed{x = 0}$

(b) $\ln(e)$

$\ln(e) = x \iff e = e^x \iff \boxed{x = 1}$

(c) $\log_6(9) + \log_6(4)$

$\log_6(9) = x_1 \iff 6^{x_1} = 9$

$\log_6(4) = x_2 \iff 6^{x_2} = 4$

$\iff 6^{x_1 + x_2} = 9 \cdot 4 = 36$

$\iff x_1 + x_2 = \log_6(36) = \boxed{2}$

Generally

$\log_b(x) + \log_b(y) = \log_b(xy)$

Exercise 6. Suppose a bacteria population doubles in size ever 30 minutes. Then, if we started out with 1000 bacteria, we can model this population by the function

$$Q(t) = 1000 \cdot (2)^{t/30}.$$

After how many hours will there be 5000 bacteria?

$5000 = 1000 \cdot (2)^{t/30}$

$5 = 2^{t/30} \iff \log_2(5) = t/30$

$\boxed{t = 30 \log_2(5)} \approx 68$