Math 135, Calculus 1, Fall 2020

09-16: Limits, Velocity, and Tangent Lines

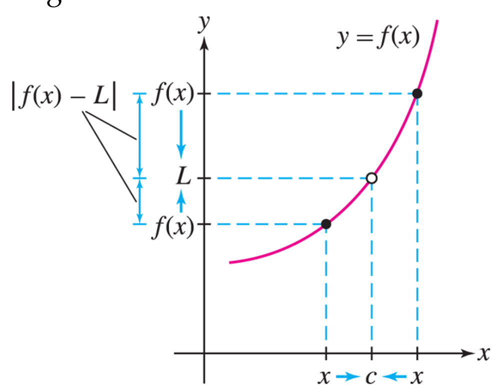
A. Limits

Definition. Given a function $f(x)$, we say that *the limit of $f(x)$ as x approaches c is equal to the number L* if $|f(x) - L|$ can be made arbitrarily small by taking x sufficiently close (but not equal) tto c.

In this case, we write

$$\lim_{x \to c} f(x) = L$$

and say "$f(x)$ approaches L as x goes to c".

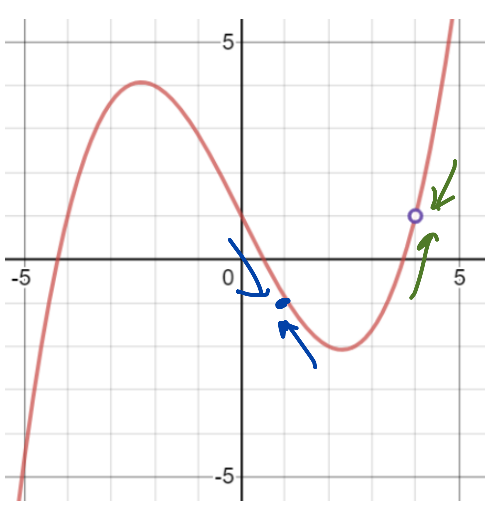


Example. On Wednesday, in Exercise 2, we numerically computed/estimated that

$$\lim_{t \to 1} \frac{t^2 + 4t - 5}{t - 1} = 6.$$

Note that, even though $\frac{t^2+4t-5}{t-1}$ is **not defined** at $t = 1$ (can't divide by 0), *the limit still exists*. We only care about the value that the function **approaches** as we get close to the point in question. The actual value of the function (or lack thereof) at the destination point is irrelevant.

Exercise 1. Using the graph of $f(x)$ below, compute $\lim_{x \to 1} f(x)$ and $\lim_{x \to 4} f(x)$.

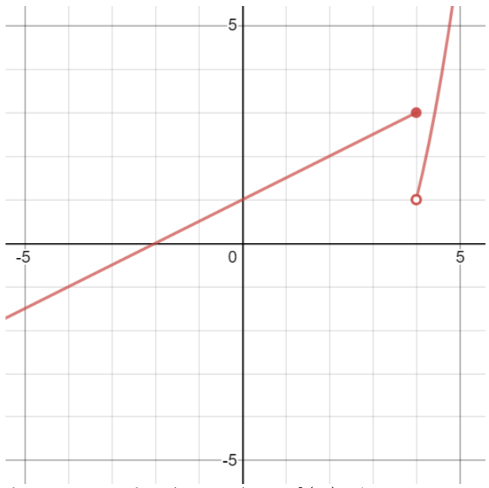


$\lim_{x \to 1} f(x) = -1$

$\lim_{x \to 4} f(x) = 1$

A.1. **One-sided limits.** Consider the function below. As x gets closer to 4, the function values approach two *different* quantities, depending on which direction x approaches 4 from:

- the **left-hand limit** at $x = 4$ (approaching from the left, $x < 4$) is 3
- the **right-hand limit** at $x = 4$ (appraoching from the right, $x > 4$) is 1.



We write

Left-hand limit: $\lim_{x\to 4^-} f(x) = 3$

Right-hand limit: $\lim_{x\to 4^+} f(x) = 1$

In this case, the limit $\lim_{x\to 4} f(x)$ **does not exist**, since the left- and right-hand limits do not agree.

$$\lim_{x\to a} f(x) = L \text{ if and only if } \lim_{x\to a^-} f(x) = L \text{ and } \lim_{x\to a^+} f(x) = L.$$

Exercise 2. Evaluate each of the following using the graph of $g(x)$ shown below.

(a) $\lim_{x\to -1^-} g(x) = -1$

(b) $\lim_{x\to -1^+} g(x) = -2$

(c) $\lim_{x\to -1} g(x)$ DNE

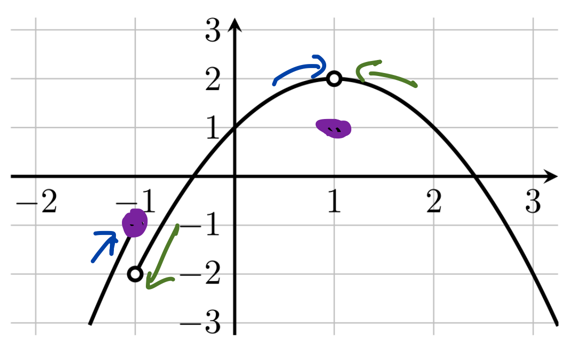
(d) $g(-1) = -1$

(e) $\lim_{x\to 1^-} g(x) = 2$

(f) $\lim_{x\to 1^+} g(x) = 2$

(g) $\lim_{x\to 1} g(x) = 2$

(h) $g(1) = 1$



A.2. **Infinite Limits.** It may happen that as x approaches the value a, the function value increases (or decreases) without bound. In these cases, we say that the limit is ∞ (or $-\infty$).

(**Warning:** These limits still **do not exist**. However, we can be precise about how they don't exist.)

Exercise 3. Compute each of the following limits. It may help to think qualitatively, e.g. "What happens as x is very close, but slightly smaller, than 3?"

(a) $\displaystyle\lim_{x\to 3^-} \frac{1}{x-3}$

• When $x>3$, $x-3>0$, so denominator is going to 0 through positive numbers, so $\to \boxed{+\infty}$

(b) $\displaystyle\lim_{x\to 3^+} \frac{1}{x-3}$

• When $x<3$, $x-3<0$, so denominator is going to 0 through negative numbers, so $\longrightarrow \boxed{-\infty}$

(c) $\displaystyle\lim_{x\to 3} \frac{1}{x-3}$

• DNE $\left(\lim_{x\to 3^-} \neq \lim_{x\to 3^+}\right)$

(d) $\displaystyle\lim_{x\to 0^-} \ln x$

• When $x<0$, $\ln(x)$ DNE, so this limit DNE.