Math 135, Calculus 1, Fall 2020

09-23: Limits and Continuity

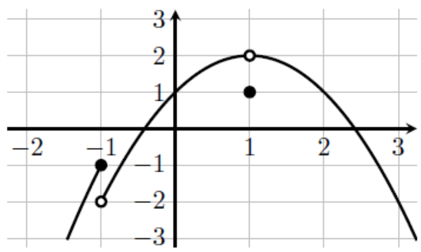
A. Verifying the solutions to 09-21

Below, we have the answer key to several problems from 09-21. Please provide the correct mathematical reasons/explainations for why these are correct.

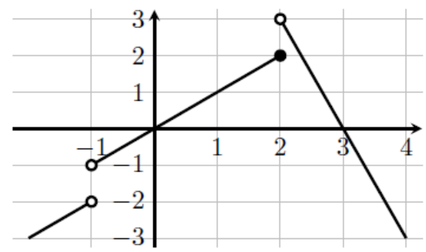
Recall that you can only apply the (numbered) Limit Laws when the limits **exist**. If any of the limits do not exist, you must use another method to determine the answer (e.g. comparing one-sided limits, calcluating infinite limits, etc).

First, consider the functions f and g from 09-21:

Graph of f



Graph of g



4. $\lim_{x\to 2^+} \left(2\, f(x) + 3\, g(x)\right) = 11$

$= 2 \lim_{x\to 2^+} f(x) + 3 \lim_{x\to 2^+} g(x) = 2\cdot 1 + 3\cdot 3 = 11$

6. $\lim_{x\to 2} \left(f(x) - g(x)\right)$ DNE.
(*Note: it is* ***not*** *sufficient to say that* $\lim_{x\to 2} g(x)$ *DNE.*)

LHS: $\lim_{x\to 2^-} f(x) - \lim_{x\to 2^-} g(x) = 1 - 2 = -1$ RHS $\lim_{x\to 2^+} f(x) - \lim_{x\to 2^+} g(x) = 1 - 3 = -2$

LHS $\neq$ RHS

8. $\lim_{x\to 3^+} \dfrac{f(x)}{g(x)} = +\infty$

$= \dfrac{-2}{0^-} = +\infty$

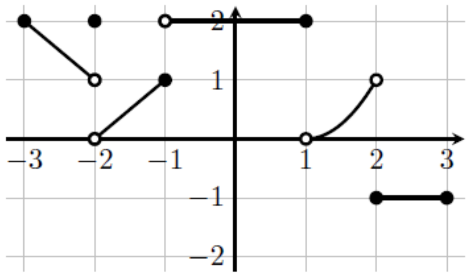
as $x \to 3^+$, $g(x) \to 0^-$ "from the left" (thru negative numbers)

11. $\lim_{x\to -1} \left(f(x) + g(x)\right) = -3$.

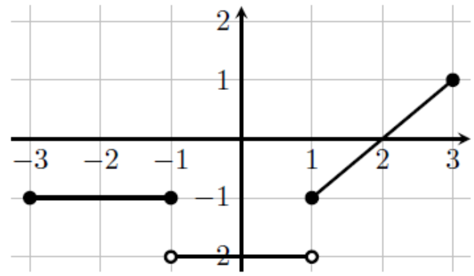
LHS: $\lim_{x\to -1^-} f(x) + \lim_{x\to -1^-} g(x) = -1 - 2 = -3$ RHS $\lim_{x\to -1^+} f(x) + \lim_{x\to -1^+} g(x) = -2 - 1 = -3$

LHS = RHS = $\boxed{-3}$

Graph of f



Graph of g



7. $\lim_{x\to 1^+} f(g(x))= 2$

as $x \to 1^+$, $g(x) \to -1^+$, so $\lim_{x\to 1^+} f(g(x)) = \lim_{x\to -1^+} f(x) = 2$

8. $\lim_{x\to -2^-} g(f(x))= -1$ (and NOT -2)
(*Note: we cannot simply say that* $\lim_{x\to -2^-} g(f(x)) = g(\lim_{x\to -2^-} f(x))$.)

as $x \to -2^-$, $f(x) \to 1^+$, so $\lim_{x\to -2^-} g(f(x)) = \lim_{x\to 1^+} g(x) = -1$

B. Continuity

In many of the limit computations from 09-21 and today, the function value and the limit values were different. This is because, in general, *the function value at $x = a$ has **no effect** on the limit as $x \to a$ of the function.*

However, some functions are better behaved:

> A function $f(x)$ is *continuous at $x = a$* if
> $$\lim_{x\to a} f(x) = f(a).$$

Intuitively:

- a function is continuous if it can be drawn without having to lift up your pencil.
- A function is *not* continuous if it has holes, jumps, asymptotes (infinite limits), or places where limits don't exist (e.g. infinite oscillations).

There are in fact *three* conditions to check to say that $f(x)$ is continuous at $x = a$:

(1) $f(a)$ must exist
(2) The limit as $x \to a$ of $f(x)$ must exist (in particular, it cannot be ∞ or $-\infty$).
(3) The limit must equal the function value.

Example B.1. Consider the function $f(x)$ on the first page. It is not continuous at $x = -1$ and $x = 1$:

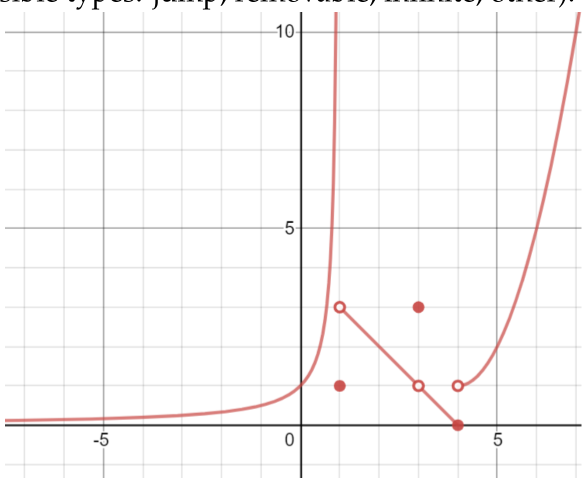
- it has a **jump discontinuity** at $x = -1$: both the left-handed and right-handed limits exist (and are finite), but they are note equal to each other.
- it has a **removable discontinuity** at $x = 2$: $\lim_{x\to 1} f(x)$ exists (and equals 2), but this is *different* from the function value $f(1) = 1$.

However, $f(x)$ is **left-continuous** at $x = -1$: the left-hand limit exists and equals the function value.

Exercise 1. Is $f(x)$ **right-continuous** at $x = 1$? Why or why not?

No: while $\lim_{x\to 2^+} f(x)$ exists, it is not equal to the function value.

Exercise 2. Consider the function $h(x)$ with the following graph, and fill in the following table (possible types: jump, removable, infinite, other).



discontinuity	type	left/right continuous?
$x = 1$	infinite	neither
$x = 3$	removable	neither
$x = 4$	jump	left

- Polynomials, rational functions, exponentials, logs, trig functions, and algebraic functions are *all continuous on their domains* [See: Limit Law Overview, "Direct Substitution Property"].
- Compositions of continuous functions are continuous.

To find limits of continuous functions, evaluate the function at the point in question (i.e. just plug it in!).

Exercise 3. Use continuity to find the value of $\lim_{t\to 3} \log_5(\cos(t-3)+4)$.

$= \log_5(\cos(3-3)+4) = \log_5(1+4) = \boxed{1}$