Math 135, Calculus 1, Fall 2020

09-25: Continuity and Limits at Infinity

A. Continuity

In many of the limit computations from 09-21 and 09-23, the function value and the limit values were different. This is because, in general, *the function value at $x = a$ has* ***no effect*** *on the limit of the function as $x \to a$.* However, some functions are better behaved:

> A function $f(x)$ is *continuous at $x = a$* if
> $$\lim_{x \to a} f(x) = f(a).$$

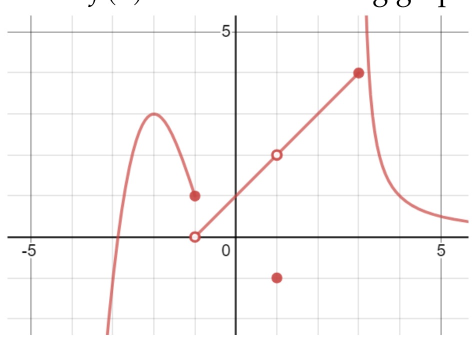
Intuitively:

- a function is continuous if it can be drawn without having to lift up your pencil.
- A function is *not* continuous if it has holes, jumps, asymptotes (infinite limits), or places where limits don't exist (e.g. infinite oscillations).

There are in fact *three* conditions to check to say that $f(x)$ is continuous at $x = a$:

(1) $f(a)$ must exist
(2) The limit as $x \to a$ of $f(x)$ must exist (in particular, it cannot be ∞ or $-\infty$).
(3) The limit must equal the function value.

Example A.1. Consider the function $f(x)$ with the following graph:



It is not continuous at $x = -1$, $x = 1$, and $x = 3$:

- it has a **jump discontinuity** at $x = -1$: both the left-handed and right-handed limits exist (and are finite), but they are note equal to each other.
- it has a **removable discontinuity** at $x = 1$: $\lim_{x \to 1} f(x)$ exists (and equals 2), but this is *different* from the function value [illegible]. $f(1) = -1$
- it has an **infinite discontinuity** at $x = 3$: (at least) one of the one-hand limits equals $\pm\infty$ (in this case $\lim_{x \to 3^+} f(x) = +\infty$).

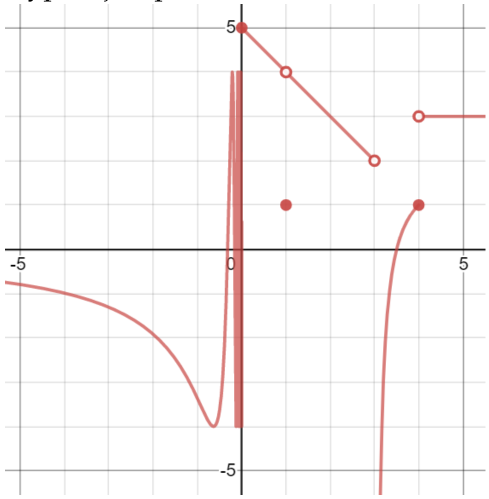
Moreover, $f(x)$ is **left-continuous** at $x = -1$: the left-hand limit exists and equals the function value.

Exercise 1. (a) Is $f(x)$ **right-continuous** at $x = 1$? Why or why not?
(b) Besides $x = -1$, where else is $f(x)$ left-continuous but not right-continuous?

(a) No: while $\lim_{x \to 1^+} f(x)$ exists, it does not equal the function value.

(b) $x = 3$: $\lim_{x \to 3^-} f(x) = 4 = f(3)$, but $\lim_{x \to 3^+} f(x) = +\infty$.

Exercise 2. Consider the function $h(x)$ with the following graph, and fill in the following table (possible types: jump, removable, infinite, other).



discontinuity	type	left/right continuous?
$x = 0$	othe	right
$x = 1$	removable	neither
$x = 3$	infinite	neither
$x = 4$	jump	Left

- Polynomials, rational functions, exponentials, logs, trig functions, and algebraic functions are *all continuous on their domains* [See: Limit Law Overview, "Direct Substitution Property"].
- Compositions of continuous functions are continuous.

To find limits of continuous functions, simply evaluate at the point in question (i.e. just plug in!).

Exercise 3. Use continuity to find the value of $\lim_{t\to 3} \log_5(\cos(t-3)+4)$.

$= \log_5(\cos(3-3)+4) = \log_5(5) = \boxed{1}$.

B. Limits at Infinity

The expression $\lim_{x\to\infty} f(x)$ means to calculate the function values of f as x gets larger and larger, and see if they approach a particular value.
As with other limits, possible answers include: a real number L, ∞, $-\infty$, or DNE.

Example B.1.

- The infinite limit $\lim_{x\to\infty} x^2 = \infty$, because as x gets larger, x^2 grows "without bound".
- The infinite limit $\lim_{x\to\infty} \dfrac{1}{x} = 0$, becuase as x gets larger, $1/x$ gets smaller and smaller.

We say that the function $f(x) = 1/x$ has a **horizontal asymptote** at $y = 0$, because the graph of f approaches the horizontal line $y = 0$ as x tends to ∞.

We can also take the limit to $-\infty$, and have horizontal asymptotes there:

Example B.2. We have

$$\lim_{x\to-\infty} x^2 = \infty, \qquad \lim_{x\to-\infty} x^3 = -\infty, \qquad \lim_{x\to-\infty} e^x = 0.$$

So $f(x) = e^x$ has a horizontal asymptote at $y = 0$.

Exercise 4. Evaluate each of the following limits, if they exist.

(a) $\lim\limits_{x\to\infty} e^{2x}$ $= \infty$

(b) $\lim\limits_{x\to\infty} e^{-2x}$ $= \lim\limits_{x\to\infty} \frac{1}{e^{2x}} = \frac{1}{\infty} = \boxed{0}$

(c) $\lim\limits_{x\to-\infty} e^{2x}$ $= \lim\limits_{x\to\infty} e^{2(-x)} = \lim\limits_{x\to\infty} e^{-2x} = \boxed{0}$

(d) $\lim\limits_{x\to\infty} -x^4 + 3x^2 + 7$ $= \lim\limits_{x\to\infty} -x^4 = \boxed{-\infty}$

(e) $\lim\limits_{x\to\infty} \sin x$ $\boxed{\text{DNE}}$ (oscillates)

(f) $\lim\limits_{x\to-\infty} 2x^2 - 3x^3$ $= \lim\limits_{x\to-\infty} -3x^3 = -(-\infty)^3 = -(-\infty) = \boxed{+\infty}$

(g) $\lim\limits_{x\to\infty} \tan^{-1} x$

· $\tan^{-1}(x) = \theta \iff \tan(\theta) = x$. So to get $x \to \infty$, need

θ in $(-\pi/2, \pi/2)$

$\left.\begin{matrix} \cos(\theta) \to 0^+ \\ \sin(\theta) \to 1 \end{matrix}\right\} \theta \to \boxed{\pi/2}$

(h) $\lim\limits_{x\to-\infty} \tan^{-1} x$ to get $x \to -\infty$, need

$\left.\begin{matrix} \cos(\theta) \to 0^+ \\ \sin(\theta) \to -1 \end{matrix}\right\} \theta \to \boxed{-\pi/2}$

Infinite limits of rational functions.

Example B.3. Consider the limit limit $\lim\limits_{x\to\infty} \frac{3x^3+5x-2}{4x^2+7}$. If we simply "plug in", we get ∞/∞, which is an **indeterminant form** (more on Monday). Instead, let's divide the top and bottom of the fraction by the **highest power in the demoninator**, which in this case is x^2. This gives:

$$\lim_{x\to\infty} \frac{3x^3+5x-2}{4x^2+7} = \lim_{x\to\infty} \frac{\frac{3x^3}{x^2} + \frac{5x}{x^2} - \frac{2}{x^2}}{\frac{4x^2}{x^2} + \frac{7}{x^2}} = \lim_{x\to\infty} \frac{3 + \frac{5}{x} - \frac{2}{x^2}}{4 + \frac{7}{x^2}} = \frac{3}{4}.$$

Exercise 5. Evaluate $\lim\limits_{x\to\infty} \frac{6x^4-5x^3+2x}{4x^2-7x^4+1}$. $= \lim\limits_{x\to\infty} \frac{6x^4}{-7x^4} = \boxed{-6/7}$

Exercise 6. Evaluate $\lim\limits_{x\to\infty} \frac{6x^4-5x^3+2x}{4x^5-7x^4+1}$ $= \lim\limits_{x\to\infty} \frac{6x^4}{4x^5} = \lim\limits_{x\to\infty} \frac{6}{4x} = \boxed{0}$

Exercise 7. Evaluate $\lim\limits_{x\to\infty} \frac{\sqrt{25x^4+10}}{3x^2+1}$. *Hint: ignore the 10 in the numerator.*

$= \lim\limits_{x\to\infty} \frac{\sqrt{25x^4}}{3x^2} = \lim\limits_{x\to\infty} \frac{5x^2}{3x^2} = \boxed{5/3}$