Math 135, Calculus 1, Fall 2020

09-28: Evaluating Limits Algebraically

A. Indeterminant forms

We have looked at several ways to evaluate the limit $\lim_{x \to a} f(x)$:

(a) If we know the function $f(x)$ is continuous at $x = a$, then the limit is simply $f(a)$.

(b) If we have the *graph* of the function f, we can visually determine the limit.

(c) We can perform numerical calculations (i.e. plug in values really, really close to a) and make a guess about the limit based on this information.

(d) We can use algebra to make the calculation easier.

Example 1. Recall the limit $\lim_{x \to 3^+} \dfrac{1}{x-3}$ considered on 09-18. Plugging in $x = 3$, we get $1/0$, which does not exist. However, it could be $+\infty$ or $-\infty$. Let's see:

(b) If we had the graph, we would see that the function values blow up as $x \to 3^+$ ("x approaches 3 from the right").

(c) Testing $x = 3.0001$, we get $f(3.0001) = 1/(0.0001) = 10000$ which just gets bigger if we add more zeros. Hence it makes sense to conclude that the limit is $+\infty$.

(d) Algebraically, we have that as $x \to 3^+$, $(x-3) \to 0^+$, so we have that the limit can be expressed as $1/0^+ = +\infty$.

Example 2. Consider the limit $\lim_{x \to -\infty} \dfrac{6x^4 - 5x^2 + 1}{3x^3 - 15}$. "Evaluating", we would get

$$\frac{6(-\infty)^4 - (-\infty)^2 + 1}{3(-\infty)^3 - 15} \text{ `` = '' } \frac{\infty - \infty}{-\infty}.$$

This expression contains two **indeterminant forms**, and thus gives us **no information**.

> The key **indeterminante forms** are $\dfrac{0}{0}$, $\dfrac{\infty}{\infty}$, $\infty \cdot 0$, and $\infty - \infty$.
>
> A limit that takes one of these forms can by *anything* (any value at all: a real number L, $+\infty$, $-\infty$, or DNE), and thus we can make **no conclusions whatsoever** about the limit based on this evaluation. Instead, *further algebraic work must be done* to find the actual value of the limit.

Example (Example 2, Continued: Technique for Infinite Limits)**.** To get rid of these indeterminant forms, we first multiply this rational function by $1 = \frac{1}{x^3} / \frac{1}{x^3}$ (where 3 is the highest power of x in the denominator) to get

$$\frac{6x^4 - 5x^2 + 1}{3x^3 - 15} = \frac{\frac{6x^4}{x^3} - \frac{5x^2}{x^3} + \frac{1}{x^3}}{\frac{3x^3}{x^3} - \frac{15}{x^3}} = \frac{6x - \frac{5}{x} + \frac{1}{x^3}}{3 - \frac{15}{x^3}}.$$

We can now evaluate this limit, and get

$$\lim_{x \to -\infty} \frac{6x^4 - 5x^2 + 1}{3x^3 - 15} = \lim_{x \to -\infty} \frac{6x - \frac{5}{x} + \frac{1}{x^3}}{3 - \frac{15}{x^3}} = \lim_{x \to -\infty} \frac{-6\infty - 0}{3 - 0} = -\infty.$$

Exercise 1. Compute $\lim\limits_{x\to-\infty} \dfrac{5x^3-2x^2+3}{-2x^2+1}$. $= \lim\limits_{x\to-\infty} \dfrac{5x^3/x^2 - 2x^2/x^2 + 3/x^2}{-2x^2/x^2 + 1/x^2}$

$$= \lim_{x\to-\infty} \frac{5x - 2 + 3/x^2}{-2 + 1/x^2} = \frac{5(-\infty) - 2 + 0}{-2+0} = \boxed{-\infty}$$

B. Changing one value

Recall the following key observation:

> The value of $f(a)$ (or even if it exists) has **no effect** on the value of $\lim\limits_{x\to a} f(x)$.

This means that if we change the function only at $x = a$, the limit is unchanged.

Exercise 2 (General technique for $\frac{0}{0}$)**.** Find the value of the limit by first canceling a common factor from the numerator and the denominator. What value of the function have we changed?

$$\lim_{x\to 2} \frac{x^2-4}{x^2+4x-12} = \frac{0}{0} \quad \text{INDETERMINANT}$$

$$\lim_{x\to 2} \frac{x^2-4}{x^2+4x-12} = \lim_{x\to 2} \frac{(x-\cancel{2})(x+2)}{(\cancel{x-2})(x+6)} = \lim_{x\to 2} \frac{x+2}{x+6} = \frac{4}{8} = \boxed{\frac{1}{2}}$$

Exercise 3. If $f(x) = 5x^2 - 3x$, find the value of $\lim\limits_{h\to 0} \dfrac{f(3+h)-f(3)}{h}$.

$$= \lim_{h\to 0} \frac{\{5(3+h)^2 - 3(3+h)\} - \{5(3)^2 - 3(3)\}}{h}$$

$$= \lim_{h\to 0} \frac{[5(9+6h+h^2) - 9 - 3h] - [45-9]}{h} = \lim_{h\to 0} \frac{5h^{\cancel{2}} + 27\cancel{h}}{\cancel{h}}$$

$$\lim_{h\to 0} 5h + 27 = \boxed{27}$$

Exercise 4 (Technique for functions with square roots)**.** Find the value of $\lim\limits_{x\to 9} \dfrac{\sqrt{x}-3}{9-x}$. (*Hint: multiply top and bottom by the conjugate* $\sqrt{x}+3$ *of the numerator.*)

$-(9-x)$

$$= \lim_{x\to 9} \frac{(\sqrt{x}-3)}{(9-x)} \frac{(\sqrt{x}+3)}{(\sqrt{x}+3)} = \lim_{x\to 9} \frac{x-9}{(\cancel{9-x})(\sqrt{x}+3)}$$

$$= \lim_{x\to 9} \frac{-1}{\sqrt{x}+3} = \frac{-1}{\sqrt{9}+3} = \boxed{\frac{-1}{6}}$$

Exercise 5. Evaluate $\lim_{\theta\to\pi/2} \frac{\tan\theta}{\sec\theta}$. $= \lim_{\theta\to\pi/2} \frac{\sin\theta/\cancel{\cos\theta}}{1/\cancel{\cos\theta}}$

$= \lim_{\theta\to\pi/2} \frac{\sin\theta}{1} = \sin(\pi/2) = \boxed{1}$

C. Other Techinques

Exercise 6 (Technique for functions with absolute values)**.** Find the value of each one-sided limits. *Hint: use the definition of the absolute value function, and the property* $|ab| = |a||b|$.

(i) $\lim_{x\to3^-} \frac{|4x-12|}{x-3}$ (ii) $\lim_{x\to3^+} \frac{|4x-12|}{x-3}$

$|x| = \begin{cases} x & x \geq 0 \\ -x & x \leq 0 \end{cases}$

$|4x-12| = \begin{cases} 4x-12 & 4x-12 \geq 0, \text{ or } x \geq 3 \\ -(4x-12) = -4x+12 & 4x-12 \leq 0, \text{ or } x \leq 3 \end{cases}$

(i) $= \lim_{x\to3^-} \frac{-4x+12}{x-3}$

$= \lim_{x\to3^-} \frac{-4\cancel{(x-3)}}{\cancel{x-3}} = \lim_{x\to3^-} -4 = \boxed{-4}$

(ii) $= \lim_{x\to3^+} \frac{4x-12}{x-3} = \lim_{x\to3^+} \frac{4\cancel{(x-3)}}{\cancel{x-3}}$

$= \lim_{x\to3^+} 4 = \boxed{4}$

Exercise 7. Find the value of $\lim_{t\to1} \frac{6}{t^2-1} - \frac{3}{t-1}$. *Hint: add the fractions and simplify.*

$= \lim_{t\to1} \frac{6}{(t-1)(t+1)} - \frac{3}{(t-1)} \cdot \frac{(t+1)}{(t+1)} = \lim_{t\to1} \frac{6-3(t+1)}{(t-1)(t+1)} = \lim_{t\to1} \frac{6-3t-3}{(t-1)(t+1)}$

$= \lim_{t\to1} \frac{-3t+3}{(t-1)(t+1)} = \lim_{t\to1} \frac{-3\cancel{(t-1)}}{\cancel{(t-1)}(t+1)} = \lim_{t\to1} \frac{-3}{t+1} = \boxed{\frac{-3}{2}}$