Math 135, Calculus 1, Fall 2020

10-02: Trig limits and the Squeeze Theorem

A. Solving limits algebraically

Last class, we reviewed the number of ways we can evaluate the limit $\lim_{x \to a} f(x)$:

- If we know the function $f(x)$ is continuous at $x = a$, then the limit is simply $f(a)$.
- If we have the *graph* of the function f, we can visually determine the limit.
- We can perform numerical calculations.
- We can use algebra.

The algebraic route is particularly useful if "evaluation" yields an **indeterminant form**:

$$\frac{0}{0}, \quad \frac{\infty}{\infty}, \quad \infty \cdot 0, \quad \infty - \infty$$

Some techniques we saw on 09-28 to deal with these:

- **Limits as x → ±∞:** only the *highest powers of x matter*. The rest we can ignore.
- **0/0** : Try canceling a common factor from both the numerator and the denominator.
  This may require you to factor polynomials, or expand functions.
- **Square roots and 0/0** : If we have square roots in the numerator or denominator, try multiplying the top and bottom by the *conjugate*.
- **∞ − ∞:** Try combining the two terms and simplifying.

B. The Squeeze Theorem

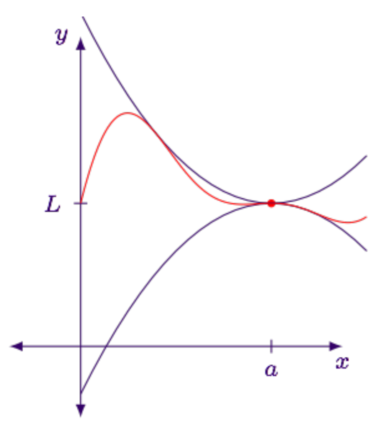
Today, we will use a new technique for when the above fail.

Theorem 1 (The Squeeze Theorem)**.** *Suppose that $f(x) \leq g(x) \leq h(x)$ when x is near a (except possibly at $x = a$), and that*

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L.$$

Then $\lim_{x \to a} g(x) = L$.

That is, if $g(x)$ is bounded above and below by two functions that limit on the same value as $x \to a$, then $g(x)$ also limits on that same value.



Example 2. Using the Squeeze Theorem, let's show that $\lim_{x\to 0} x^2 \sin(1/x) = 0$. Evaluating, we get $0 \cdot DNE$, which is unhelpful. However, we know the range of $\sin\theta$ is just $[-1, 1]$, so this gives us bounds for $\sin(1/x)$:

$$-1 \le \sin(1/x) \le 1$$

Multiplying through by $x^2 > 0$ does not change the inequality signs, so we get

$$-x^2 \le x^2 \sin(1/x) \le x^2$$

But now $\lim_{x\to 0} -x^2 = \lim_{x\to 0} x^2 = 0$, so the limit of the bounded function $\lim_{x\to 0} x^2 \sin(1/x) = 0$.

Exercise 1. Use the Squeeze Theorem to compute $\lim_{x\to\infty} \dfrac{\sin x}{x}$.

$-1 \le \sin(x) \le 1$ for all x

divide by x

$-1/x \le \sin(x)/x \le 1/x$

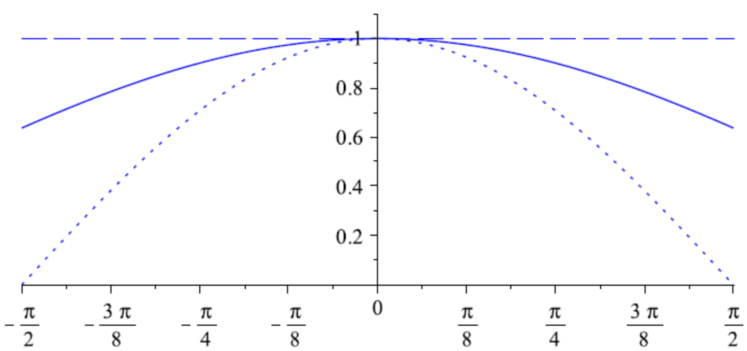
$\lim_{x\to\infty} -1/x = 0 = \lim_{x\to\infty} 1/x$, so Squeeze Thm $\Rightarrow$ $\boxed{\lim_{x\to\infty} \frac{\sin(x)}{x} = 0}$

Two important trig limits are the following:

$$\boxed{\lim_{x\to 0} \frac{\sin x}{x} = 1 \qquad \text{and} \qquad \lim_{x\to 0} \frac{1-\cos x}{x} = 0.}$$

Example 3. To prove the first one, we use the fact that

$$\cos x \le \frac{\sin x}{x} \le 1 \qquad \text{for } \tfrac{-\pi}{2} < x < \tfrac{\pi}{2}$$



- Solid line: $y = \dfrac{\sin x}{x}$
- Dashed line: $y = 1$
- Dotted line: $y = \cos x$

Since $\lim_{x\to 0} \cos x = \lim_{x\to 0} 1 = 1$, the Squeeze Theorem implies $\lim_{x\to 0} \dfrac{\sin x}{x} = 1$, as desired.

Exercise 2. Using the limit $\lim_{x\to 0} \frac{\sin x}{x} = 1$, show that $\lim_{x\to 0} \frac{1-\cos x}{x} = 0$.
Hint: Multiply the top and bottom by $1+\cos x$*; simplify; and then break the fraction into the product of two fractions, one of which is* $\frac{\sin x}{x}$.

$$\lim_{x\to 0} \frac{1-\cos(x)}{x} \cdot \frac{(1+\cos(x))}{(1+\cos(x))} = \lim_{x\to 0} \frac{1-\cos^2(x)}{x(1+\cos x)} = \lim_{x\to 0} \frac{\sin^2 x}{x(1+\cos x)}$$

$$= \lim_{x\to 0} \left[\frac{\sin x}{x} \cdot \frac{\sin x}{1+\cos x}\right] = \left(\lim_{x\to 0} \frac{\sin x}{x}\right) \cdot \left(\lim_{x\to 0} \frac{\sin x}{1+\cos x}\right)$$

$$= 1 \cdot \left(\frac{0}{1+1}\right) = 1 \cdot 0 = \boxed{0}$$

Exercise 3. Evaluate $\lim_{x\to 0} \frac{\sin^2 x}{x^2}$.
Hint: the limit of a product equals the product of the limits.

$$= \lim_{x\to 0} \left[\frac{\sin x}{x} \cdot \frac{\sin x}{x}\right] = \left(\lim_{x\to 0} \frac{\sin x}{x}\right) \cdot \left(\lim_{x\to 0} \frac{\sin x}{x}\right)$$

$$= 1 \cdot 1 = \boxed{1}$$

Exercise 4. Evaluate $\lim_{t\to 0} \frac{\sin(7t)}{t}$.
Hint: Make the substitution $x = 7t$. *If* $t \to 0$, *what is* x *approaching? Use this substitution to rewrite the limit using only the variable* x *in a way so that* $\frac{\sin x}{x}$ *is present.*

- $t \to 0 \Rightarrow 7t = x \to 7 \cdot 0 = 0$
- $\sin(7t) = \sin(x)$
- $t = x/7$

$$\Rightarrow \lim_{t\to 0} \frac{\sin(7t)}{7}$$

$$= \lim_{x\to 0} \frac{\sin(x)}{x/7}$$

$$= \lim_{x\to 0} 7 \cdot \frac{\sin(x)}{x}$$

$$= 7 \cdot \lim_{x\to 0} \frac{\sin x}{x}$$

$$= 7 \cdot 1 = \boxed{7}$$