Math 135, Calculus 1, Fall 2020

10-05: Definition of the Derivative (Section 3.1)

This section introduces the **derivative**, one of the most important concepts in all of mathematics and science. It is the foundation of calculus.

What is the derivative? We have already seen examples of the derivative in our earlier work: the derivative of $f(x)$ is

- the **slope** of the tangent line to the graph of the function $f(x)$, or
- the **instantaneous velocity** of $f(x)$ at a point.

We compute this by taking the limit of the average velocities/slopes of secant lines: if we fix one endpoint $x = a$, then

$$\text{average velocity on the interval } [a, x] = \text{slope of the secant line over } [a, x] = \frac{\Delta y}{\Delta x} = \frac{f(x) - f(a)}{x - a}$$

Taking the limit, we get the following:

Definition. The **derivative** of $f(x)$ at the point $x = a$, denoted by $f'(a)$ (read as "f prime of a"), is given by

$$\boxed{f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}. \qquad (1)}$$

This limit may or may not exist. If the limit exists, we say that the function is **differentiable** at $x = a$.

The fraction in Equation (1) is called the **difference quotient**. If we evaluate the difference quotient at $x = a$ we get $\frac{0}{0}$, a familiar indeterminant form that can be ANYTHING. So we must simplify the difference quotient in order to evaluate the limit.

Exercise 1. Use Equation (1) to compute the derivative of $f(x) = x^2 - 3x$ at the point $x = 1$. In other words, compute $f'(1)$. What is the equation of the tangent line to f at $x = 1$?
Hint: What is the value of a? What is f(a)? Can we factor the numerator?
Be careful with your parentheses!

$a = 1$

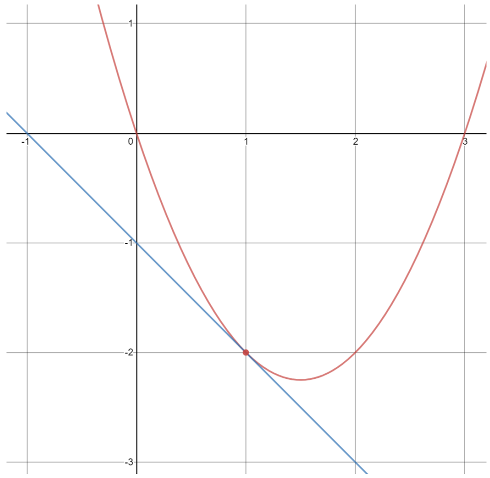
$f(a) = (1)^2 - 3(1) = 1 - 3 = -2$

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{x \to 1} \frac{[x^2 - 3x] - [-2]}{x - 1}$$

$$= \lim_{x \to 1} \frac{x^2 - 3x + 2}{x - 1} = \lim_{x \to 1} \frac{\cancel{(x-1)}(x-2)}{\cancel{(x-1)}} = \lim_{x \to 1} (x - 2)$$

$$= 1 - 2 = \boxed{-1}$$

Compare your answer to the above with the following graph of $y = x^2 - 3x$ with the tangent line at $x = 1$.



We can rewrite the definition from Equation (1) by making the substitution $h = x - a$. Then we have the average velocity on the interval $[a, x] = [a, a + h]$ is the difference quotient

$$\frac{f(x) - f(a)}{x - a} = \frac{f(a + h) - f(a)}{h}.$$

This has a simpler denominator, but a more complicated numerator. Since $h \to 0$ as $x \to a$, we have a second definition of the derivative:

$$f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}. \tag{2}$$

Exercise 2. Use Equation (2) to compute $f'(1)$ for $f(x) = x^2 - 3x$. Confirm that you obtain the same ansqewr as in Exercise 1.
Hint: What is the value of a? What is $f(a + h)$? Can we factor the numerator?
Be careful with your parentheses!

- $a = 1$
- $f(a+h) = f(1+h) = (1+h)^2 - 3(1+h) = 1 + 2h + h^2 - 3 - 3h$
  $= h^2 - h - 2$
- $f(a) = -2$
- $\displaystyle\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0} \frac{(h^2 - h - 2) - (-2)}{h}$

$\displaystyle = \lim_{h \to 0} \frac{h^{\cancel{2}} - \cancel{h}}{\cancel{h}} = \lim_{h \to 0} h - 1 = 0 - 1 = \boxed{-1}$ ✓

Exercise 3. Consider the function $f(x) = -4x + 7$. What do you expect for the value of $f'(3)$? Why? Confirm your guess using either Equation (1) or (2).

Guess: -4, as the tangent line to a line is simply the line itself

$$\lim_{x \to 3} \frac{f(x) - f(3)}{x - 3} = \lim_{x \to 3} \frac{(-4x+7) - (-4(3)+7)}{x - 3}$$

$$= \lim_{x \to 3} \frac{-4x + 12}{x - 3} = \lim_{x \to 3} \frac{-4\cancel{(x-3)}}{\cancel{x-3}} = \lim_{x \to 3} (-4) = \boxed{-4} \checkmark$$

Exercise 4. Using either Equation (1) or (2), find the slope of the tangent line to the function $f(x) = \dfrac{2}{x} + 1$ at the point $x = 2$.

Hint: Can we write the numerator as a single fraction?

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Exercise 5. Let $f(x) = \sqrt{x}$. Compute $f'(4)$ using either Equation (1) or (2).

Hint: Multiply the top and bottom of the difference quotient by the conjugate of the numerator.