Math 135, Calculus 1, Fall 2020

10-07: The Derivative, Graphs, and First Rules

Recall the **derivative function** $f'(x)$ of a function $f(x)$ is defined to be

- the slope of the tangent line
- the instantaneous velocity
- $\lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h}$

A. Graphs of $f'(x)$

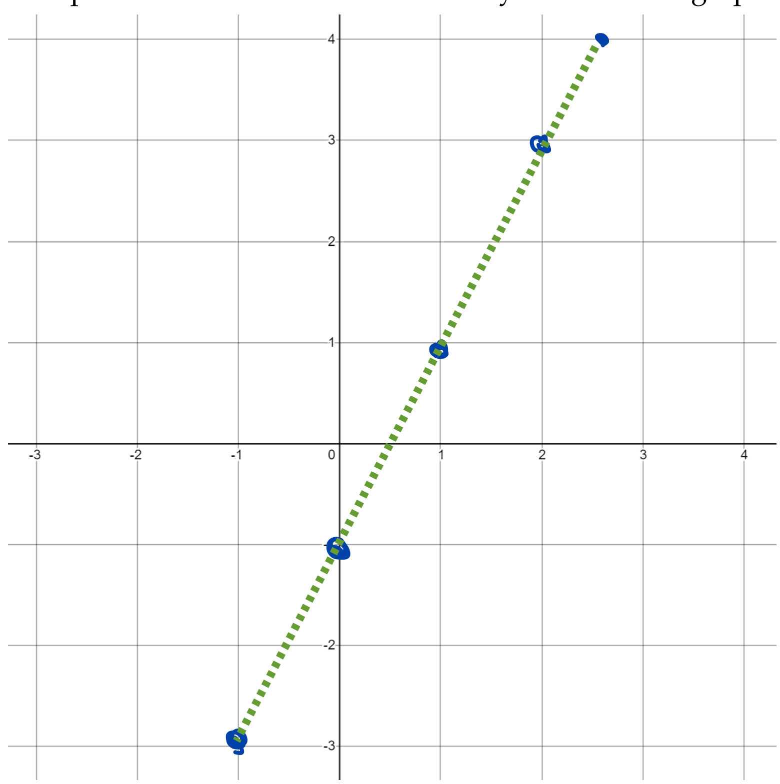
Exercise 1. Consider the function $f(x) = x^2 - x - 2$. We will construct the graph of the derivative function one point at a time.

(a) Go to the website http://www.shodor.org/interactivate/activities/Derivate/.

(b) Enter the function $y = f(x)$ above. Use the tool to cacluate the slope of the tangent line at each of the points $x = -1,\ 0,\ 1,\ 2,$ and 2.5. Enter these values in the table below:

x	-1	0	1	2	2.5
slope of g at x	-3	-1	1	3	4

(c) Now plot these points and connect them smoothly to estiamte a graph of $f'(x)$.



(d) What do you think the formula for this graph is?

$y = 2x - 1$

Exercise 2. Use the limit definition of the derivative to compute $f'(x)$ for $f(x) = x^2 - x - 2$. Do your results match your results from Exercise 1?

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{[(x+h)^2 - (x+h) - 2] - [x^2 - x - 2]}{h}$$

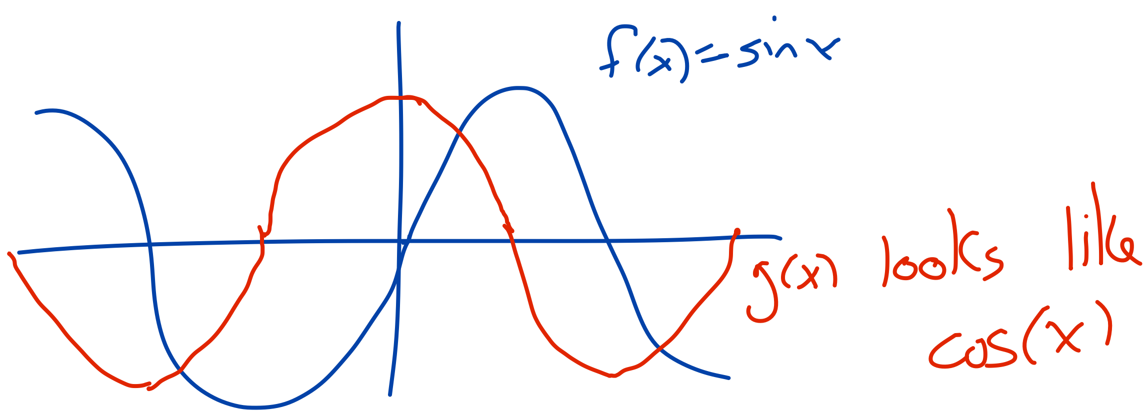
$$= \lim_{h\to 0} \frac{[\cancel{x^2} + 2xh + h^2 \cancel{- x} - h \cancel{- 2}] \cancel{- x^2} \cancel{+ x} \cancel{+ 2}}{h}$$

$$= \lim_{h\to 0} \frac{2x\cancel{h} + h^{\cancel{2}\,1} - \cancel{h}^{\,1}}{\cancel{h}} = \lim_{h\to 0} (2x + h - 1) = \boxed{2x - 1}$$

Exercise 3. Desmos offers an interactive applet that gives a good visual and tactile experience of poroducing the derivative function point by point. Follow the link below and follow the instructions, using the function $f(x) = \sin x$.

https://www.desmos.com/calculator/jlcpl1spy2

In the space below, sketch the graphs of $f(x)$ and $f'(x)$. Do you recognize the graph of $f'(x)$?



B. Differentiability

If the limit defining $f'(a)$ exists, we say f is **differentiable** at $x = a$.

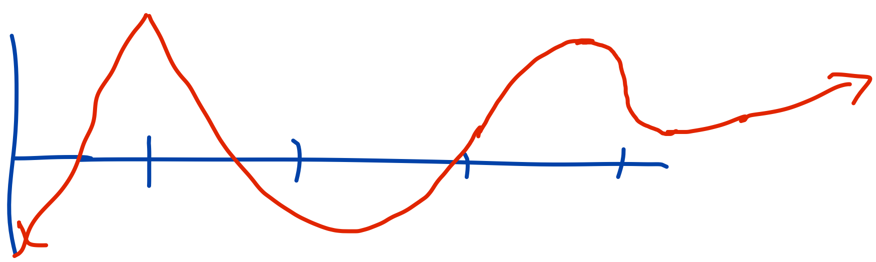
Theorem 1. *If $f(x)$ is differentiable at $x = a$, then it is also continuous there. However, the converse is **not** true: a function may be continuous at a point, but not differentiable there (e.g. $f(x) = |x|$ is continuous at $x = 0$, but is not differentiable there).*

What might go wrong? Any of the things that make a limit not exist. In paticular:

(a) the left-hand limit might not equal the right-hand limit.
(b) the limit might be infinite.

Exercise 4. Sketch the graph of a continuous function that is differentiable everywhere **except**:

- at $x = 1$ because the limit DNE as in (a) above (the derivative from the left does not equal the derivative from the right)
- at $x = 4$ because the limit DNE as in (b) above (the derivative is infinite)



Notation 2. The derivative of a function $f(x)$ has a second notation, namely

$$f'(x) = \frac{dy}{dx} \qquad\qquad f'(a) = \left.\frac{dy}{dx}\right|_{x=a}$$

This alterantive notation $\frac{dy}{dx}$ is read as "the derivative of y with respect to x", and is called **Leibniz notation**. It reminds us that the derivative is the slope:

$$m \equiv \frac{\Delta y}{\Delta x} \qquad \text{so} \qquad f'(x) = \frac{dy}{dx}$$

Leibniz notation also helps to represent taking the derivatives as an operation: the symbol $\frac{d}{dx}$ means "take the derivative with respect to x".

Exercise 5. Use the limit definition of the derivative to prove that the derivative of a line is its slope:

$\frac{d}{dx}(mx + b) =$ $\lim_{h\to 0} \frac{[m(x+h)+b]-[mx+b]}{h}$

$$= \lim_{h\to 0} \frac{\cancel{mx}+mh+\cancel{b}-\cancel{mx}-\cancel{b}}{h} = \lim_{h\to 0} \frac{m\cancel{h}}{\cancel{h}} = \lim_{h\to 0} m = \boxed{m}$$