Math 135, Calculus 1, Fall 2020

10-12: Derivative Rules 1

Last week, we introduced the **derivative function** $f'(x)$ of a function $f(x)$, whose evaluation $f'(a)$ at the point $x = a$ is give by:

- the slope of the tangent line at $x = a$
- the instantaneous velocity at time $x = a$

In general, the rule of the derivative function $f'(x)$ can be computed as the limit

$$f'(x) = \lim_{h \to x} \frac{f(x+h) - f(x)}{h}$$

where the numerator is (and remains) a function of x. However, this **limit definition of the derivative** can be cumbersome. If only there were some **rules** or **patterns** we could find, that would help us not need to go through the elaborate limit calculations.

We saw the first of these last Friday: the derivative when $f(x) = mx + b$ is a line is just the *constant function* $f'(x) = m$ at the slope of the line. This was proved using the limit definition of the derivative, but now we never need to use it again for a line.

A. "There's got to be a better way!"

The following formulas can all be computed using the limit definition of the derivative. However, they are **general**, and can then be used when computing the derivative of many different functions.

- **Constant Rule:** $\dfrac{d}{dx}(c) = 0$
- **Power Rule:** $\dfrac{d}{dx}(x^n) = nx^{n-1}$ for **any** real number n.
- **Constant Multiples:** $\dfrac{d}{dx}\big(c \cdot f(x)\big) = c \cdot f'(x)$ for any constant c
- **Sums and Differences:** $\dfrac{d}{dx}\big(f(x) \pm g(x)\big) = f'(x) \pm g'(x)$
- **Exponential Functions:** $\dfrac{d}{dx}(b^x) = \ln b \cdot b^x$, so in particular $\dfrac{d}{dx}(e^x) = e^x$.

Exercise 1. Verify the Power Rule when $n = 3$; that is, use the limit definition of the derivative to show that $\dfrac{d}{dx}(x^3) = 3x^2$.

$$\lim_{h \to 0} \frac{\left[(x+h)^3\right] - \left[x^3\right]}{h} = \lim_{h \to 0} \frac{\cancel{x^3} + 3x^2h + 3xh^2 + h^3 - \cancel{x^3}}{h}$$

$$= \lim_{h \to 0} \frac{3x^2\cancel{h} + 3xh^{\cancel{2}} + h^{\cancel{3}\,2}}{\cancel{h}} = \lim_{h \to 0}\left(3x^2 + 3xh + h^2\right) = \boxed{3x^2} \checkmark$$

Exercise 2. Find each of the following derivatives using the power rule:

(a) $\frac{d}{dx}(x^{15})$ (b) $\frac{d}{dx}\left(\frac{1}{x^4}\right)$ (c) $\frac{d}{dx}(\sqrt{x})$ (d) $\frac{d}{dx}(x^{\pi})$

$\boxed{15x^{14}}$

$= \frac{d}{dx}(x^{-4}) = -4x^{-5} = \boxed{-4/x^5}$

$= \frac{d}{dx}(x^{1/2}) = \frac{1}{2}x^{-1/2} = \boxed{\frac{1}{2\sqrt{x}}}$

$= \boxed{\pi \cdot x^{\pi-1}}$

Exercise 3. Find $g'(x)$ if $g(x) = 6\sqrt{x} - \frac{3}{x^3} + 5e^x - \pi^4$.

$g(x) = 6x^{1/2} - 3x^{-3} + 5e^x - \pi^4$

$g'(x) = 6 \cdot \frac{1}{2}x^{-1/2} - 3(-3)x^{-4} + 5 \cdot e^x - 0$

$\boxed{g'(x) = \frac{3}{\sqrt{x}} + \frac{9}{x^4} + 5e^x}$

Exercise 4. If $p(x) = 4\sqrt[3]{x} + \frac{2}{3}x - \frac{8}{x}$, find the equation of the tangent line to p at the point $x = 8$.

- $p(x) = 4x^{1/3} + \frac{2}{3}x - 8x^{-1} \implies p'(x) = \frac{4}{3}x^{-2/3} + \frac{2}{3} + 8x^{-2}$
- slope $= p'(8) = \frac{4}{3}(8)^{-2/3} + \frac{2}{3} + 8 \cdot 8^{-2} = \frac{4}{3} \cdot \frac{1}{2^2} + \frac{2}{3} \cdot \frac{8}{8^2}$

  $= \frac{1}{3} + \frac{2}{3} + \frac{1}{8} = \boxed{9/8}$
- point $= (8, p(8)) = (8, 4 \cdot \sqrt[3]{8} + \frac{2}{3} \cdot 8 - \frac{8}{8}) = (8, 8 + \frac{16}{3} - 1) = \boxed{(8, \frac{32}{3})}$

$\boxed{y = f'(a)(x-a) + f(a) = 9/8(x-8) + 32/3}$

Exercise 5. Use the limit definition of the derivative to verify that $\frac{d}{dx}(e^x) = e^x$. You may use the following: $\lim_{x \to 0} \frac{e^x - 1}{x} = \cancel{x}\ 1$

$\lim_{h \to 0} \frac{e^{(x+h)} - e^x}{h} = \lim_{h \to 0} \frac{e^x \cdot e^h - e^x}{h} = \lim_{h \to 0} e^x \cdot \left(\frac{e^h - 1}{h}\right)$

$= e^x \cdot \left(\lim_{h \to 0} \frac{e^h - 1}{h}\right) = e^x \cdot 1 = \boxed{e^x}$

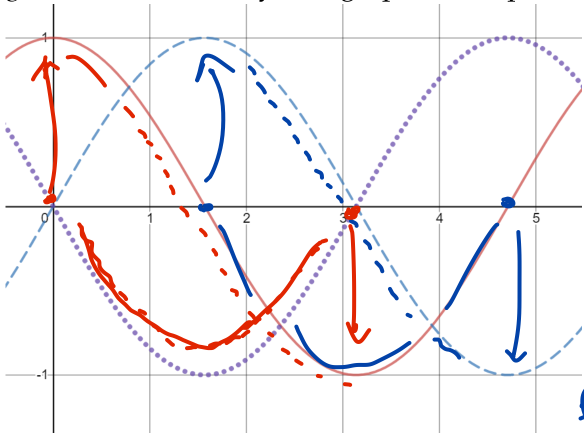
Exercise 6. If the graph of $g(t)$ is a parabola, what type of graph will $g'(t)$ be? Explain.

a line – a parabola has function of the form $g(t) = at^2 + bt + c$, so $g'(t) = 2at + b$ is a line.

Exercise 7. If $z = e^t + t^e$, find $\frac{dz}{dt}$.

$$\frac{dz}{dt} = e^t + e \cdot t^{e-1}$$

Exercise 8. The graph belows shows three functions: $f(x)$, $g(x)$, and $h(x)$. If $f'(x) = g(x)$ and $g'(x) = h(x)$, identify that graph that represents each function. Explain.



derivative 0 $\iff$ function flat

derivative > 0 $\iff$ function increasing

derivative < 0 $\iff$ function decreasing

Looks like:

$f(x)$ = blue dashed,

$f'(x) = g(x)$ = red

$g'(x) = h(x)$ = purple dots