Math 135, Calculus 1, Fall 2020

10-14: Product of Quotient Rules

Last week, we introduced the **derivative function** $f'(x)$ of a function $f(x)$, whose evaluation $f'(a)$ at the point $x = a$ is give by:

- the slope of the tangent line at $x = a$
- the instantaneous velocity at time $x = a$

On Monday, we covered algorithms to help us compute the derivatives of polynomials and exponential functions. Today, we'll tackle **products** and **quotients**

Theorem (Product Rule)**.** *If $f(x)$ and $g(x)$ are differentiable functions, then so is their product $f(x) \cdot g(x)$. The derivative of the product is given by*

$$\boxed{\frac{d}{dx}\Big(f(x) \cdot g(x)\Big) = f'(x) \cdot g(x) + f(x) \cdot g'(x).}$$

Example 1 (Warning)**.** The derivative of a product is **not** equal to the product of the derivatives:

Consider the product $x \cdot x$. Then

$$\frac{d}{dx}(x) \cdot \frac{d}{dx}(x) = 1 \cdot 1 = 1$$

but instead

$$\frac{d}{dx}(x \cdot x) = \frac{d}{dx}(x) \cdot x + x \cdot \frac{d}{dx}(x) = 1 \cdot x + x \cdot 1 = 2x$$

which is what the **power rule** gives us for the derivative of $x \cdot x = x^2$.

Exercise 1. Use the Product Rule to find $f'(x)$ when $f(x) = (3x^2 + 1)e^x$. Simlify your answer.

$$\frac{d}{dx}\left[(3x^2+1)\cdot(e^x)\right] = \frac{d}{dx}(3x^2+1)\cdot(e^x) + (3x^2+1)\cdot\frac{d}{dx}(e^x)$$

$$=(6x)(e^x)+(3x^2+1)(e^x) = \boxed{(3x^2+6x+1)(e^x)}$$

Theorem (Quotient Rule)**.** *If $f(x)$ and $g(x)$ are differentiable functions, then so is their quotient $f(x)/g(x)$ whenever $g(x) \neq 0$. The derivative of the quotient is given by*

$$\boxed{\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}.}$$

Remark 2. The Quotient Rule can be derived from the product rule: Letting $Q(x) = \dfrac{f(x)}{g(x)}$, cross multiply the equation and differentiate both sides with respect to x using the Product Rule. Solving for $Q'(x)$ leads to the above formula.

Exercise 2. Use the Quotient Rule to calculate the derivative of $\frac{1}{x^4}$ and check your answer against the result obtained from using the Power Rule.

- top $= 1$, $\frac{d}{dx}(\text{top}) = 0$
- bottom $= x^4$, $\frac{d}{dx}(\text{bottom}) = 4x^3$

$$\frac{d}{dx}\left(\frac{1}{x^4}\right) = \frac{\frac{d}{dx}(\text{top})\cdot\text{bottom} - \text{top}\cdot\frac{d}{dx}(\text{bottom})}{(\text{bottom})^2} = \frac{0\cdot x^4 - 1\cdot 4x^3}{(x^4)^2}$$

$$= \frac{-4x^3}{x^8} = \boxed{-4/x^5}$$

$$\frac{d}{dx}(x^{-4}) = \boxed{-4x^{-5}}$$

equal!

Exercise 3. If $g(x) = \frac{3x+1}{2x-5}$, find a simplify $g'(x)$.

$$g'(x) = \frac{\frac{d}{dx}(3x+1)(2x-5) - (3x+1)\cdot\frac{d}{dx}(2x-5)}{(2x-5)^2}$$

$$= \frac{(3)(2x-5) - (3x+1)(2)}{(2x-5)^2} = \frac{6x-15-6x-2}{(2x-5)^2} = \boxed{\frac{-17}{(2x-5)^2}}$$

Exercise 4. If $h(x) = \frac{e^x}{x^2+1}$, find and simplify $h'(x)$.

$$h'(x) = \frac{\frac{d}{dx}(e^x)(x^2+1) - (e^x)\cdot\frac{d}{dx}(x^2+1)}{(x^2+1)^2} = \frac{e^x(x^2+1) - e^x(2x)}{(x^2+1)^2}$$

$$= \boxed{\frac{e^x(x^2-2x+1)}{(x^2+1)^2}}$$

Exercise 5. Suppose that $f(3) = 5$, $f'(3) = -7$, $g(3) = 2$, and $g'(3) = 1/2$. If $H(x) = \frac{f(x)}{xg(x)}$, find $H'(3)$.

$$H'(x) = \frac{f'(x)\cdot[x\cdot g(x)] - f(x)\cdot\frac{d}{dx}[x\cdot g(x)]}{(x\cdot g(x))^2}$$

$$= \frac{f'(x)\cdot g(x)\cdot x - f(x)(1\cdot g(x) + x\cdot g'(x))}{(x\cdot g(x))^2}$$

$$H'(3) = \frac{f'(3)\cdot g(3)\cdot 3 - f(3)(g(3) + (3)\cdot g'(3))}{(3\cdot g(3))^2} = \frac{(-7)(2)(3) - (5)\left(2 + 3\cdot\left(\frac{1}{2}\right)\right)}{(3\cdot 2)^2}$$

$$= \frac{\left(-42 - \frac{35}{2}\right)}{36} = \boxed{\frac{-119}{72}}$$