Math 135, Calculus 1, Fall 2020

10-16: Higher Order Derivatives (Section 3.5) and Trig Derivatives (Section 3.6)

Last week, we introduced the **derivative function** $f'(x)$ of a function $f(x)$, whose evaluation $f'(a)$ at the point $x = a$ is give by:

- the slope of the tangent line at $x = a$
- the instantaneous velocity at time $x = a$
- the instantaneous rate of change of f with respect to x

Today, we'll be discussing **higher order derivatives** and the derivatives of **trig functions**.

A. Higher Derivatives

The derivative $f'(x)$ of a function $f(x)$ gives the **slope** of the function f at the point x. However, $f'(x)$ is also a function, and so we can ask: "What is the derivative of the derivative?"

Definition 1. The **second derivative** of $f(x)$ is the function

$$\boxed{f''(x) = \frac{d}{dx}\big(f'(x)\big) = \frac{d^2 f}{dx^2} = \frac{d^2 y}{dx^2}.}$$

To compute the second derivative, we simply take the derivative twice.

Exercise 1. Suppose that $f(x) = 2x^4 - 3x + e^x$. Find $f'(x)$ and $f''(x)$.

$f'(x) = 8x^3 - 3 + e^x$

$f''(x) = 24x^2 + e^x$

A.1. **The second derivative and concavity.** The second derivative measures the change in $f'(x)$, i.e. the change in the slope of $f(x)$. What does this really mean?

Recall:

- $\frac{d}{dx}(f)|_{x=a} > 0 \quad \Leftrightarrow \quad$ the slope is positive at $x = a \quad \Leftrightarrow \quad f(x)$ is **increasing** at $x = a$
- $\frac{d}{dx}(f)|_{x=a} < 0 \quad \Leftrightarrow \quad$ the slope is negative at $x = a \quad \Leftrightarrow \quad f(x)$ is **decreasing** at $x = a$

If $f''(a) > 0$, then $\frac{d}{dx}(f')|_{x=a} > 0$, so the slopes of f are increasing at $x = a$. There are two options:

- If the slopes are already positive, then they are getting bigger, so the curve is getting steeper, increasing at a faster rate (like e^x)
- If the slopes are negative, then the function $f(x)$ is still decreasing, but beginning to flatten out: the negative slopes are increasing towards (and possibly past) zero.

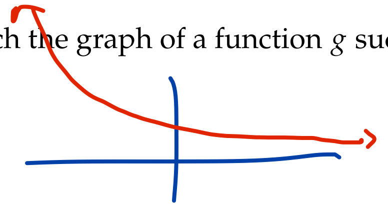
In these cases, we say the graph of f is **concave up** at $x = a$.

Similarly, the reverse options can happen if $f''(a) < 0$, and the graph is **concave down**.

Concave up:

Concave down:

Exercise 2. Sketch the graph of a function g such that $g'(x) < 0$ and $g''(x) > 0$ everywhere.



$y = e^{-x}$

Exercise 3. Former President Nixon famously said, "Although the rate of inflation is increasing, it is increasing at a decreasing rate." Let $r(t)$ denote the rate of inflation. According to President Nixon, what are the signs (+, −, or 0) of $r'(t)$ and $r''(t)$?

$r'(t) > 0, \quad r''(t) < 0$

A.2. **Higher Derivatives.** .

In good cases, we can continue to take derivatives of derivatives.

- We write $f'''(x)$ for the **third derivative** $\dfrac{d}{dx}(f''(x))$.
- More generally, we write $f^{(n)}(x)$ for the n**-th derivative** of $f(x)$.

Exercise 4. Suppose that $f(x) = xe^x$.

(a) Find and simplify $f'(x)$, $f''(x)$, and $f'''(x)$.

$f'(x) = e^x + xe^x = \boxed{(1+x)e^x}$, $f''(x) = e^x + (1+x)e^x = \boxed{(2+x)e^x}$,

$f'''(x) = e^x + (2+x)e^x = \boxed{(3+x)e^x}$

(b) Find a general formula, in terms of n, of the n-th derivative $f^{(n)}(x)$.

$$\boxed{f^{(n)}(x) = (n+x)e^x}$$

Exercise 5. The inflation rate is given by the (positive) rate of change of the **consumer price index**. Let $p(t)$ denote the consumer price index. According to Nixon, what are the signs (+, −, or 0) of $p'(t)$, $p''(t)$, and $p'''(t)$?

$p'(t) = r(t)$, so $p'(t) > 0$, $p''(t) > 0$, $p'''(t) < 0$

Exercise 6. A news report out of Massachusetts yesterday said:

> The total number of COVID cases that were confirmed last week grew to 4,560 today. That's a 12% increase over the previous week and an 83% increase in cases over the week of Sept. 13, when cases began to rise at a higher rate.

Let $N(t)$ denote the cumulative total number of cases of COVID-19 in Massachusetts. What derivative of N went from negative to positive on September 13? Using evidence from the news article, is that derivative still positive?

- $N''(t)$ went from negative to positive ("rate of new cases" $= N'(t)$ started increasing)
- $N'(t)$ is bigger this week than last week, so $N''(t)$ is still positive.

B. Trig Derivatives

Using the trig identities $\sin(A+B) = \sin A \cdot \cos B + \cos A \sin B$, along with the trig limits $\lim_{x\to 0} \frac{\sin x}{x} = 1$ and $\lim_{x\to 0} \frac{1-\cos x}{x} = 0$, we can compute the derivatives of $\sin(x)$ and $\cos(x)$:

Theorem 2. *If x is measured in radians, then*

$$\frac{d}{dx}(\sin x) = \cos x, \qquad \frac{d}{dx}(\cos x) = -\sin x.$$

To prove the first formula, let $f(x) = \sin x$. Then

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{\sin(x+h) - \sin(x)}{h} \\ &= \lim_{h\to 0} \frac{\sin x \cdot \cos h + \cos x \cdot \sin h - \sin x}{h} \\ &= \lim_{h\to 0} \frac{\sin x \cdot (\cos h - 1) + \cos x \cdot \sin h}{h} \\ &= \lim_{h\to 0} \sin x \cdot \frac{\cos h - 1}{h} \quad + \quad \lim_{h\to 0} \cos x \cdot \frac{\sin h}{h} \\ &= \sin x \cdot 0 + \cos x \cdot 1 \\ &= \cos x. \end{aligned}$$

Exercise 7. Use the quotient rule and the above results to prove that $\frac{d}{dx}(\tan x) = \sec^2 x$.

$$\frac{d}{dx}(\tan x) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) = \frac{\cos x \cdot \cos x - \sin x \cdot (-\sin x)}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x \ \checkmark$$

Exercise 8. Show that $\frac{d}{dx}(\sec x) = \sec x \cdot \tan x$.

$$\frac{d}{dx}\left(\frac{1}{\cos x}\right) = \frac{\frac{d}{dx}(1) \cdot \cos x - 1 \cdot \frac{d}{dx}(\cos x)}{\cos^2 x} = \frac{0 - (-\sin x)}{\cos^2 x}$$

$$= \frac{\sin x}{\cos^2 x} = \frac{1}{\cos x} \cdot \frac{\sin x}{\cos x} = \sec x \cdot \tan x \ \checkmark$$

Exercise 9. If $g(x) = x^3 \sin x$, find a simplify $g'(x)$ and $g''(x)$.

$$g'(x) = \frac{d}{dx}(x^3) \cdot \sin x + (x^3) \cdot \frac{d}{dx}(\sin x) = \boxed{3x^2 \cdot \sin x + x^3 \cdot \cos x}$$

$$g''(x) = \left[\frac{d}{dx}(3x^2) \cdot \sin x + (3x^2) \frac{d}{dx}(\sin x)\right] + \left[\frac{d}{dx}(x^3) \cdot \cos x + (x^3) \cdot \frac{d}{dx}(\cos x)\right]$$

$$= 6x \cdot \sin x + 3x^2 \cos x + 3x^2 \cos x - x^3 \sin x = \boxed{\sin x\,(6x - x^3) + \cos x\,(6x^2)}$$