Math 135, Calculus 1, Fall 2020

10-19: The Chain Rule (Section 3.7)

Last week, we introduced the **derivative function** $f'(x)$ of a function $f(x)$, whose evaluation $f'(a)$ at the point $x = a$ is give by:

- the slope of the tangent line at $x = a$
- the instantaneous velocity at time $x = a$
- the instantaneous rate of change of f with respect to x

Today: Trig derivatives and the chain rule.

A. Trig Derivatives

Theorem 1. *If x is measured in radians, then*

$$\frac{d}{dx}(\sin x) = \cos x, \qquad \frac{d}{dx}(\cos x) = -\sin x.$$

The proof can be found in Worksheet 10-16.

Exercise 1. Use the quotient rule and the above results to prove that $\frac{d}{dx}(\cot x) = -\csc^2 x$.

$$\frac{d}{dx}\left(\frac{\cos x}{\sin x}\right) = \frac{\frac{d}{dx}(\cos x)\sin x - \cos x \cdot \frac{d}{dx}(\sin x)}{\sin^2 x}$$

$$= \frac{-\sin x \cdot \sin x - \cos x \cdot \cos x}{\sin^2 x} = \frac{-(\underbrace{\sin^2 x + \cos^2 x}_{1})}{\sin^2 x}$$

$$= \frac{-1}{\sin^2 x}$$

$$= \boxed{-\csc^2 x} \checkmark$$

B. Chain Rule

The chain rule gives us a way to compute the derivative of a **composite** of two functions.

Theorem 2. *If $O(x)$ and $I(x)$ are differentiable functions, then so is the composite $O(I(x)) = (O \circ I)(x)$. Moreover,*

$$\boxed{\frac{d}{dx}\Big(O(I(x))\Big) = O'\big(I(x)\big) \cdot I'(x).}$$

That is, "the derivative of the outside function **evaluated at the inside function**, times the derivative of the inside function".

Example 3. Let $h(x) = (x^4 + 1)^2$. The **inside function** is $I(x) = x^4 + 1$, as this is the **first** thing we do to evaluate this function. The **outside function** is $O(x) = x^2$, as this is the **second** thing we do to evaluate this function. Then $h(x) = O(I(x))$.

Since $O'(x) = 2x$ and $I'(x) = 4x^3$, we have $O'(I(x)) = 2 \cdot (x^4 + 1)$, and the Chain Rule says that

$$h'(x) = O'(I(x)) \cdot I'(x) = 2(x^4 + 1) \cdot 4x^3 = 8x^7 + 8x^3.$$

To check this, we can first exapnd $h(x)$ and compute $h'(x)$ via the Power Rule:

$$h(x) = (x^4 + 1)^2 = x^8 + 2x^4 + 1, \qquad h'(x) = 8x^7 + 8x^3.$$

Exercise 2. Let $f(x) = \sin(2x)$.

(a) Use the double-angle formula $\sin(2x) = 2\sin x \cos x$ to compute $f'(x)$ using the product rule.

$$\frac{d}{dx}\left(2\sin x \cos x\right) = \frac{d}{dx}(2\sin x)\cdot \cos x + (2\sin x)\frac{d}{dx}(\cos x)$$

$$= 2\cos x \cos x - 2\sin x \sin x = 2\underbrace{(\cos^2 x - \sin^2 x)}_{\text{double angle formula}}$$

$$= \boxed{2\cos(2x)}$$

(b) Use the Chain Rule to directly compute $f'(x)$. (*What is the inside function? Outside function?*)

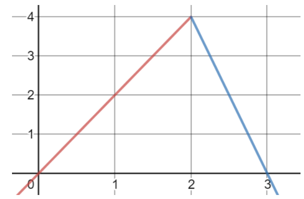
inside: $2x$

outside: $\sin(x)$

$$\frac{d}{dx}\left(\sin(2x)\right) = \cos(2x)\cdot\frac{d}{dx}(2x) = \boxed{\cos(2x)\cdot 2} \checkmark$$

Exercise 3. Let $f(x)$ and $g(x)$ be two functions. Certain values of $f(x)$ and $f'(x)$ are given in the table below, and the graph of $g(x)$ is as shown.

x	1	2	3
f(x)	3	2	1
f′(x)	4	5	6



(a) Let $h(x) = g(f(x))$. Find $h'(3)$.

Computation. The Chain Rule says that $h'(x) = g'(f(x)) \cdot f'(x)$, so

$$h'(3) = g'(f(3)) \cdot f'(3) = g'(1) \cdot 6 = 2 \cdot 6 = 12.$$

□

(b) Let $k(x) = f(g(x))$. Find $k'(1)$.

$$k'(x) = f'(g(x)) \cdot g'(x)$$

$$k'(1) = f'(g(1)) \cdot g'(1) = f'(2) \cdot 2 = 5 \cdot 2 = \boxed{10}$$

Exercise 4. If $F(x) = \sqrt{x^4 + 3}$, use the Chain Rule to find and simplify $F'(x)$.

$$F'(x) = \frac{1}{2\sqrt{x^4+3}} \cdot \frac{d}{dx}(x^4 + 3)$$

$$= \frac{4x^3}{2\sqrt{x^4+3}}$$

$$= \boxed{\frac{2x^3}{\sqrt{x^4+3}}}$$

Exercise 5. (a) If $y = e^{x^2}$, compute $\frac{dy}{dx}$.

outside : $y = e^t$
inside : $t = x^2$

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = (e^t)\cdot(2x) = \boxed{e^{x^2}\cdot 2x}$$

(b) If $z = e^{\tan t}$, compute $\frac{dz}{dt}$.

outside : e^x
inside : $\tan(t)$

$$\frac{dz}{dt} = e^{(\tan t)}\cdot\frac{d}{dt}(\tan t) = \boxed{\sec^2 t\cdot e^{\tan t}}$$

You may need to use the Chain Rule multiple times:

Exercise 6. Find and simplify $G'(x)$ if $G(x) = \sin\left(\sqrt{x^2+2}\right) + e^{\cos(4x)}$.

$$G'(x) = \cos\left(\sqrt{x^2+2}\right)\cdot\frac{d}{dx}\left(\sqrt{x^2+2}\right) + e^{\cos(4x)}\cdot\frac{d}{dx}\left(\cos(4x)\right)$$

$$= \cos\left(\sqrt{x^2+2}\right)\cdot\frac{1}{2\sqrt{x^2+2}}\frac{d}{dx}(x^2+2)$$

$$+ e^{\cos(4x)}\cdot(-\sin(4x))\cdot\frac{d}{dx}(4x)$$

$$= \boxed{\frac{\cos\left(\sqrt{x^2+2}\right)\cdot\cancel{2}x}{\cancel{2}\sqrt{x^2+2}} - 4\sin(4x)e^{\cos(4x)}}$$