Math 135, Calculus 1, Fall 2020

10-21: The Chain Rule, Part II (Section 3.7)

Last week, we introduced the **derivative function** $f'(x)$ of a function $f(x)$, whose evaluation $f'(a)$ at the point $x = a$ is give by:

- the slope of the tangent line at $x = a$
- the instantaneous velocity at time $x = a$
- the instantaneous rate of change of f with respect to x

Today: More with the chain rule.

A. Chain Rule

The chain rule gives us a way to compute the derivative of a **composite** of two functions.

Theorem 1. *If $O(x)$ and $I(x)$ are differentiable functions, then so is the composite $O(I(x)) = (O \circ I)(x)$. Moreover,*

$$\boxed{\frac{d}{dx}\Big(O(I(x))\Big) = O'\big(I(x)\big) \cdot I'(x).}$$

Exercise 1. Compute the derivative of $f(x) = \sin(x^2)$.

Exercise 2. Suppose that $h(x) = f(g(x))$, and that $f'(3) = 4$, $f(3) = 2$, $f'(6) = -1$, $g(3) = 6$, $g'(3) = 7$, $g(2) = 4$, and $g'(2) = 11$. Find $h'(3)$.

Leibniz Notation. If $y = f(x)$ and $x = g(t)$, then y is a function of t by $y = f(g(t))$. By the Chain Rule, we have that $\frac{dy}{dt} = f'(g(t)) \cdot g'(t)$, or more succinctly,

$$\boxed{\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}.}$$

When computing the derivative of the inside function, you may need to use any of the derivative rules we have discussed so far: the product rule, quotient rule, or even the chain rule itself.

Exercise 3. Compute $\frac{dy}{dt}$ when $y = \sqrt{\frac{t+1}{t-1}}$.

$$\left(\frac{1}{2\sqrt{t+1/t-1}}\right) \cdot \frac{d}{dt}\left(\frac{t+1}{t-1}\right)$$

Exercise 4. Compute $\frac{d}{dt}\Big(\tan(\cos(e^{6t}))\Big)$.

$$-6e^{6t} \cdot \sec^2\left(\cos(e^{6t})\right) \cdot \sin\left(e^{t}\right)$$

$$\sec^2\left(\cos(e^{6t})\right) \cdot -\sin\left(e^{6t}\right) \cdot e^{6t} \cdot 6$$

Exercise 5. Find and simplify $F'(x)$ if $F(x) = \sin(x^2)\cos(x^2)$.