Math 135, Calculus 1, Fall 2020

10-22: Implicit Differentiation (Section 3.8)

The **derivative** $f'(x)$ of a function $f(x)$ gives:

- the slope of the tangent line
- the instantaneous velocity
- the instantaneous rate of change

A. Chain Rule

The chain rule gives us a way to compute the derivative of a **composite** of two functions.

Theorem. *If $O(x)$ and $I(x)$ are differentiable functions, then so is the composite $O(I(x)) = (O \circ I)(x)$. Moreover,*

$$\boxed{\frac{d}{dx}\Big(O(I(x))\Big) = O'\big(I(x)\big) \cdot I'(x).}$$

Exercise 1. Compute $\frac{d}{dx}\left(\sin\left(e^{\sqrt{2x}}\right)\right)$.

$$= \cos\left(e^{\sqrt{2x}}\right) \cdot \frac{d}{dx}\left(e^{\sqrt{2x}}\right)$$

$$= \cos\left(e^{\sqrt{2x}}\right) \cdot e^{\sqrt{2x}} \cdot \frac{d}{dx}\left(\sqrt{2x}\right)$$

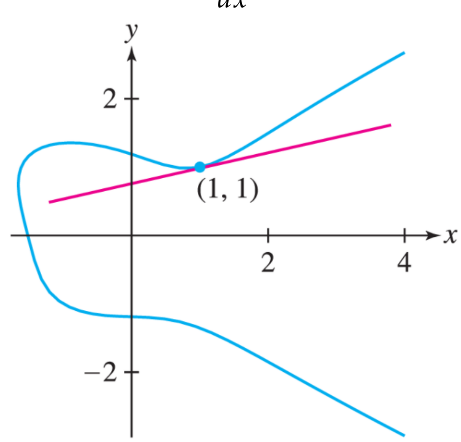
$$= \cos\left(e^{\sqrt{2x}}\right) \cdot e^{\sqrt{2x}} \cdot \frac{1}{2\sqrt{2x}} \cdot \frac{d}{dx}(2x) = \boxed{\frac{\cos\left(e^{\sqrt{2x}}\right) \cdot e^{\sqrt{2x}}}{\sqrt{2x}}}$$

B. Implicit Differentiation

If y and x are related not by a function, but by a general equation such as

$$y^4 + xy = x^3 - x + 2,$$

we should still be able to compute the function $\frac{dy}{dx}$, the slope of the tangent line at a point.



Example 1. To compute $\frac{dy}{dx}$ for $x +^2 +y^2 = 2x$, we take the derivative of both sides of the equation with respect to x:

$$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(2x)$$
$$2x + \frac{d}{dx}(y^2) = 2$$

To compute $\frac{d}{dx}(y^2)$, we use the Chain Rule: we think of y as representing a function $y = y(x)$ of x, and then the chain rule says

$$\frac{d}{dx}(y^2) = \frac{d}{dx}(y(x)^2) = 2 \cdot y(x) \cdot y'(x) = 2y\frac{dy}{dx}.$$

All together, we have

$$2x + 2y\frac{dy}{dx} = 2,$$

and solving for $\frac{dy}{dx}$ (and assuming $y \neq 0$) we get

$$\frac{dy}{dx} = \frac{1-x}{y}.$$

Exercise 2. Use the Product Rule to compute $\frac{d}{dx}(xy)$.

$$= \frac{d}{dx}(x) \cdot y + x \cdot \frac{d}{dx}(y)$$

$$= \boxed{y + x\frac{dy}{dx}}$$

Exercise 3. Use the Quotient Rule to compute $\frac{d}{dx}\left(\frac{y}{x}\right)$.

$$= \frac{\frac{d}{dx}(y) \cdot x - y \cdot \frac{d}{dx}(x)}{(x)^2}$$

$$= \boxed{\frac{dy/dx \cdot x - y}{x^2}}$$

Exercise 4. Compute $\frac{dy}{dx}$ if $\sin(xy) = y^2$.

$$\frac{d}{dx}\left(\sin(xy)\right) = \frac{d}{dx}\left(y^2\right)$$

$$\cos(xy)\cdot\frac{d}{dx}(xy) = 2\cdot y\cdot\frac{d}{dx}(y)$$

$$\cos(xy)\left[y + x\frac{dy}{dx}\right] = 2y\frac{dy}{dx}$$

$$y\cdot\cos(xy) + x\cdot\cos(xy)\cdot\frac{dy}{dx} = 2y\cdot\frac{dy}{dx}$$

$$\boxed{\frac{dy}{dx} = \frac{y\cdot\cos(xy)}{2y - x\cdot\cos(xy)}}$$

Exercise 5. Compute $\frac{dy}{dx}$ if $e^x + e^y = xy$.

$$\frac{d}{dx}\left(e^x + e^y\right) = \frac{d}{dx}(xy)$$

$$\frac{d}{dx}(e^x) + \frac{d}{dx}(e^y) = \left[y + x\frac{dy}{dx}\right]$$

$$e^x + e^y\cdot\frac{dy}{dx} = y + x\frac{dy}{dx}$$

$$\boxed{\frac{dy}{dx} = \frac{e^x - y}{x - e^y}}$$

Exercise 6. Find the slope of the tangent line to the graph of $e^y\sin(x) = 1$ at $(\pi/2, 0)$.

$$\frac{d}{dx}\left(e^y\cdot\sin(x)\right) = \frac{d}{dx}(1)$$

$$\frac{d}{dx}(e^y)\cdot\sin(x) + e^y\cdot\frac{d}{dx}(\sin x) = 0$$

$$e^y\cdot\frac{dy}{dx}\cdot\sin(x) + e^y\cos x = 0$$

$$\boxed{\frac{dy}{dx} = \frac{-e^y\cos x}{e^y\cdot\sin x} = \frac{-\cos x}{\sin x}}$$

$$\left.\frac{dy}{dx}\right|_{(\pi/2,0)} = \frac{-\cos(\pi/2)}{\sin(\pi/2)} = \frac{-0}{1} = \boxed{0}$$