Math 135, Calculus 1, Fall 2020

10-22: Implicit Differentiation (Section 3.8)

The **derivative** $f'(x)$ of a function $f(x)$ gives:

- the slope of the tangent line
- the instantaneous velocity
- the instantaneous rate of change

A. Implicit Differentiation

If y and x are related not by a function, but by a general equation

$$F(x, y) = k$$

we can compute the derivative $\frac{dy}{dx}$, the slope of the tangent line at a point, using **implicit differentiation**.

Exercise 1. Compute $\frac{dy}{dx}$ if $x^2 - y^2 + 2xy = 5$.

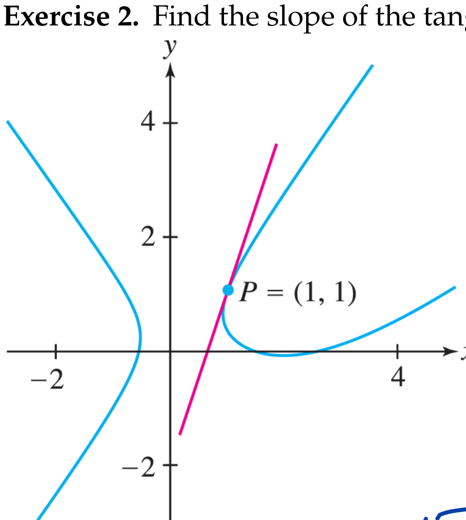
$$\frac{d}{dx}(x^2) - \frac{d}{dx}(y^2) + \frac{d}{dx}(2xy) = \frac{d}{dx}(5)$$

$$2x - 2y\frac{dy}{dx} + \left(\frac{d}{dx}(2x)\cdot y + (2x)\cdot\frac{d}{dx}(y)\right) = 0$$

$$2x - 2y\frac{dy}{dx} + 2y + 2x\frac{dy}{dx} = 0$$

$$\boxed{\frac{dy}{dx} = \frac{-2x-2y}{2x-2y}}$$

Exercise 2. Find the slope of the tangent line to $e^{y-x} = 2y^2 - x^2$ at the point $(1, 1)$.



$$\frac{d}{dx}(e^{y-x}) = \frac{d}{dx}(2y^2) - \frac{d}{dx}(x^2)$$

$$e^{y-x}\cdot\frac{d}{dx}(y-x) = 4y\frac{dy}{dx} - 2x$$

$$e^{y-x}\left(\frac{dy}{dx} - 1\right) = 4y\frac{dy}{dx} - 2x$$

$$\boxed{\frac{dy}{dx} = \frac{e^{y-x} - 2x}{e^{y-x} - 4y}}$$

$$\left.\frac{dy}{dx}\right|_{\substack{x=1\\y=1}} = \frac{e^0 - 2(1)}{e^0 - 4(1)} = \frac{1-2}{1-4} = \boxed{\frac{1}{3}}$$

B. Derivative of $\ln(x)$

We can use implicit differentiation to compute the derivatives of **inverse functions**. Recall that a function $f(x)$ is **1-to-1** if each function value $y = f(a)$ is hit exactly one time. In this case, there is an inverse function $f^{-1}(x)$ such that

$$f^{-1}\big(f(x)\big) = x$$

or equivalently that the left equation holds exactly when the right equation holds:

$$f(a) = b \qquad a = f^{-1}(b)$$

Example 1. The function $f(x) = e^x$ and $f^{-1}(x) = \ln(x)$ are inverse functions. More generally, b^x and $\log_b(x)$ are inverse functions.

Exercise 3. Compute the derivative of $y = \ln(x)$:

(i) Consider the equivalent equation $e^y = x$, and find $\frac{dy}{dx}$ using implicit differentiation.

$$\frac{d}{dx}(e^y) = \frac{d}{dx}(x)$$

$$e^y \cdot \frac{dy}{dx} = 1$$

$$\boxed{\frac{dy}{dx} = 1/e^y}$$

(ii) Substitute x back in (*what is x equal to?*), so that the resulting function for $\frac{dy}{dx}$ is only in terms of x (no y's allowed!).

$$x = e^y, \text{ so } \frac{dy}{dx} = 1/e^y = \boxed{1/x}$$

Exercise 4. If $p(x) = \ln(x^2 + 1)$, find $p'(x)$.

$$p'(x) = \frac{1}{x^2+1} \cdot \underbrace{\frac{d}{dx}(x^2+1)}_{\text{chain rule term}} = \boxed{\frac{2x}{x^2+1}}$$

C. Derivative of inverse trig functions

In this section, we will follow a similar pattern as in Section B. However, to "substitute x back in", we will need to use a *geometric* understanding of the trig functions.

Exercise 5. Compute the derivative of $y = \sin^{-1}(x)$:

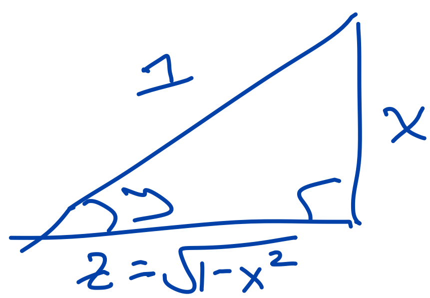
(i) Consider the equivalent equation $\sin(y) = x$, and find $\frac{dy}{dx}$ using implicit differentiation.

$$\frac{d}{dx}(\sin(y)) = \frac{d}{dx}(x)$$

$$\cos(y) \cdot \frac{dy}{dx} = 1$$

$$\boxed{\frac{dy}{dx} = \frac{1}{\cos(y)}}$$

(ii) Draw a triangle with angle y, and label the sides in terms of x.



- $\sin(y) = \frac{\text{opp}}{\text{hyp}} = \frac{x}{1}$
- $z^2 + x^2 = 1$

$$z = \sqrt{1-x^2}$$

(iii) Substitute x back in (no y's allowed!).

$$\cos(y) = \frac{\text{adj}}{\text{hyp}} = \sqrt{1-x^2}$$

$$\boxed{\frac{dy}{dx} = \frac{1}{\cos(y)} = \frac{1}{\sqrt{1-x^2}}}$$