Math 135, Calculus 1, Fall 2020

10-30: Logarithmic Differentiation (Section 3.8)

The **derivative** $f'(x)$ of a function $f(x)$ gives:

- the slope of the tangent line
- the instantaneous velocity
- the instantaneous rate of change

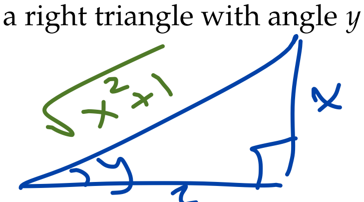
A. Derivatives of inverse trig functions

Exercise 1. Use implicit differentiation to find the derivative of $y = \arctan(x)$:

(a) Use implicit differentiation to find $\frac{dy}{dx}$ for the equivalent equation $tan(y) = x$.

$$\sec^2(y)\cdot\frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = \frac{1}{\sec^2(y)} = \cos^2(y)$$

(b) Draw a right triangle with angle y and $\tan(y) = x$.



$$a^2+b^2=c^2$$
$$1^2+x^2=c^2$$
$$c=\sqrt{x^2+1}$$

(c) Use the above triangle to remove all instances of y in your expression for $\frac{dy}{dx}$ from Part (a).

$$\cos(y) = \frac{adj}{hyp} = \frac{1}{\sqrt{x^2+1}} \Rightarrow \frac{dy}{dx} = \left(\frac{1}{\sqrt{x^2+1}}\right)^2 = \boxed{\frac{1}{x^2+1}}$$

B. Derivative of b^x and $\log_b(x)$

Let $b > 0$. Since e^x and $ln(x)$ are inverse functions, we know in particular that

$$e^{\ln(f(x))} = f(x) \tag{1}$$

for any function $f(x)$.

Exercise 2. Use Equation (1) and the Chain Rule to compute $f'(x)$ for $f(x) = b^x$.

- $f(x) = e^{\ln(b^x)} = e^{\ln(b)\cdot x}$
- $f'(x) = e^{\ln(b)\cdot x}\cdot\ln(b) = e^{\ln(b^x)}\cdot\ln(b) = \boxed{b^x\cdot\ln(b)}$

Exercise 3. Use implicit differentiation (as in Exercise 1) to compute $\frac{d}{dx}\Big(\log_b(x)\Big)$.

$$y = \log_b(x) \Longleftrightarrow b^y = x$$

$$\frac{d}{dx}\Big(\quad b^y\cdot\ln(b)\cdot\frac{dy}{dx} = 1$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{b^y\ln(b)} = \boxed{\frac{1}{x\cdot\ln(b)}}$$

C. Logarithmic Differentiation

Logarithmic differentiation is a technique which can be used to turn a tedious derivative calculuation, involving lots of Product and Quotient Rules, into a relatively easy procedure.

Exercise 4. Consider the function $f(x) = \dfrac{x(x+1)^3}{(3x-1)^2}$. Gross.

(a) Take the natural log of both sides of this equation, and simplify the right-hand-side by using log rules:

$$\ln(a \cdot b) = \ln(a) + \ln(b), \qquad \ln\left(\frac{a}{b}\right) = \ln(a) - \ln(b), \qquad \ln\left(a^b\right) = b \cdot \ln(a).$$

$$\ln(f(x)) = \ln\left(\frac{x(x+1)^3}{(3x-1)^2}\right) = \ln\left(x(x+1)^3\right) - \ln\left((3x-1)^2\right)$$

$$= \ln(x) + \ln\left((x+1)^3\right) - 2\ln(3x-1) = \boxed{\ln(x) + 3\ln(x+1) - 2\ln(3x-1)}$$

d/dx

(b) Use implicit differentiation to compute $f'(x)$ in terms of x and $f(x)$.

$$\frac{1}{f(x)} \cdot \frac{d}{dx}(f(x)) = \frac{1}{x} + \frac{3}{x+1} \cdot \frac{d}{dx}(x+1) - \frac{2}{3x-1}\,\frac{d}{dx}(3x-1)$$

$$\frac{f'(x)}{f(x)} = \frac{1}{x} + \frac{3}{x+1} - \frac{6}{3x-1} \Rightarrow \boxed{f'(x) = \left(\frac{1}{x} + \frac{3}{x+1} - \frac{6}{3x-1}\right) \cdot f(x)}$$

(c) Replace $f(x)$ with the original expression to find $f'(x)$ just in terms of x.

$$\boxed{f'(x) = \left(\frac{1}{x} + \frac{3}{x+1} - \frac{6}{3x-1}\right)\left(\frac{x(x+1)^3}{(3x-1)^2}\right)}$$

Logarithmic differentiation is also useful to compute the derivatives of functions of the form $y = f(x)^{g(x)}$.

Exercise 5. Using the same method as in Exercise 4, compute $f'(x)$ if $f(x) = \sin(x)^{\cos(x)}$.

$$\ln(f(x)) = \ln\left(\sin(x)^{\cos(x)}\right) = \cos(x) \cdot \ln(\sin(x))$$

$\frac{d}{dx}$

$$\frac{1}{f(x)} \cdot f'(x) = \frac{d}{dx}(\cos x)\ln(\sin(x)) + \cos(x) \cdot \frac{d}{dx}\left(\ln(\sin(x))\right)$$

$$= -\sin x \ln(\sin x) + \cos x \cdot \frac{1}{\sin x} \cdot \frac{d}{dx}(\sin x)$$

$$= -\sin x \cdot \ln(\sin x) + \cos^2 x / \sin x$$

$$f'(x) = \left\{-\sin x \cdot \ln(\sin x) + \cos^2 x / \sin x\right\} \cdot f(x) = \boxed{\left\{-\sin x \ln(\sin x) + \frac{\cos^2 x}{\sin x}\right\} \cdot \sin(x)^{\cos(x)}}$$