Math 135, Calculus 1, Fall 2020

11-02: Logarithmic Differentiation (Section 3.8) and Rates of Change (Section 3.4)

The **derivative** $f'(x)$ of a function $y = f(x)$ gives:

- the slope of the tangent line
- the instantaneous rate of change of y with respect to x

A. Logarithmic Differentiation

Exercise 1. Consider the function $f(x) = (3x-2)^{(7x^2+1)}\cancel{x}$.

(a) Take the natural log of both sides of this equation, and simplify the right-hand-side by using log rules:

$$\ln(a \cdot b) = \ln(a) + \ln(b), \qquad \ln\left(\frac{a}{b}\right) = \ln(a) - \ln(b), \qquad \ln\left(a^b\right) = b \cdot \ln(a).$$

$$\ln(f(x)) = \ln\left[(3x-2)^{(7x^2+1)}\right] = (7x^2+1) \cdot \ln(3x-2)$$

(b) Use implicit differentiation to compute $f'(x)$ in terms of x and $f(x)$.

$$\frac{1}{f(x)} \frac{d}{dx}(f(x)) = \frac{d}{dx}(7x^2+1) \cdot \ln(3x-2) + (7x^2+1) \frac{d}{dx}(\ln(3x-2))$$

$$\frac{f'(x)}{f(x)} = 14x \ln(3x-2) + (7x^2+1) \cdot \frac{3}{3x-2}$$

$$\boxed{f'(x) = \left[14x \cdot \ln(3x-2) + \frac{21x^2+3}{3x-2}\right] \cdot f(x)}$$

(c) Replace $f(x)$ with the original expression to find $f'(x)$ just in terms of x.

$$\boxed{f'(x) = \left[\ln\left((3x-2)^{14x}\right) + \frac{21x^2+3}{3x-2}\right]\left((3x-2)^{(7x^2+1)}\right)}$$

B. Rates of Change

For any function $y = f(x)$, the derviative $\dfrac{dy}{dx}$ measures the **instantaneous rate of change** of y with respect to x.

Example 1. If $T(t)$ measures the temperature T (in degrees Celsius) of an object as a function of time t (in seconds), then $\dfrac{dT}{dt}$ measures the rate the temperature of the object is changing. The units of $\frac{dT}{dt}$ are $^\circ C/\text{sec}$. If $\frac{dT}{dt} > 0$, then the object is warming; if $\frac{dT}{dt} < 0$, then the object is cooling.

B.1. Application to Economics.

Let $C(x)$ be the cost of producing a quantity x of some item, e.g. $C(25) = \$3000$ means it costs \$3000 to produce 25 items. The derivative $C'(x)$ is called the **marginal cost**, and gives an approximation to the cost of producing the $(x+1)$-st item. Similarly:

- if $P(x)$ is the profit made from selling x items, then $P'(x)$ is called the **marginal profit**, and
- if $R(x)$ is the revenue made from selling x items, then $R'(x)$ is called the **marginal revenue**.

Exercise 2. Suppose $C(x) = 8000 - 10x + x^2 + 0.01x^3$ represents the cost of producing x computers.

(a) Find the marginal cost function.

$$C'(x) = -10 + 2x + 0.03x^2$$

(b) Find $C'(10)$ and explain its meaing. What are the units of $C'(10)$?

$$C'(10) = -10 + 2(10) + 0.03(100)$$
$$= -10 + 20 + 3 = \boxed{13 \text{ \$/computer}}$$

- This means the cost to produce the 11th computer is approx \$13.

(c) Find the actual cost of producing the 11th computer. Compare your answer with $C'(10)$.

$$C(11) - C(10) = \left(8000 - 10(11) + (11)^2 + (0.01)(11)^3\right)$$
$$- \left(8000 - 10(10) + (10)^2 + (0.01)(10)^3\right)$$
$$= 24.31 - 10$$
$$= \boxed{\$14.31}$$

B.2. **Application to Physics.** If $s(t)$ is the position of a moving object as a function of time t, then $s'(t) = v(t)$ is the **instantaneous velocity**, and $s''(t) = v'(t) = a(t)$ is the **instantaneous acceleration**. The speed of the object is defined to be $|s'(t)| = |v(t)|$, which is always positive.

Exercise 3. Suppose a particle moves according to the equation $s(t) = t^3 - 12t^2 + 36t$ for $t \geq 0$, where s, the position, is measured in meters, and t, the time, is measured in seconds.
Think of the particle moving along a number line, with s indicating the position on the line.

(a) Compute the velocity and acceleration of the particle at time t.

$$v(t) = s'(t) = 3t^2 - 24t + 36$$

$$a(t) = v'(t) = s''(t) = 6t - 24$$

(b) When is the particle at rest?

When is $v(t) = 0$?

$$0 = 3t^2 - 24t + 36$$

$$0 = t^2 - 8t + 12 = (t-6)(t-2)$$

$\boxed{t = 6}$ $\boxed{t = 2}$

(c) What is the particle moving to the right? to the left?

Right? $v(t) > 0$ $\boxed{(0,2) \cup (6, \infty)}$

Left? $v'(t) < 0$ $\boxed{(2,6)}$

	0		2		6	
$v(t)$		$+$	0	$-$	0	$+$

$v(1) = 3 - 24 + 36 > 0$
$v(3) = 3 \cdot 9 - 24 \cdot 3 + 36 < 0$
$v(10) = 300 - 240 + 36 > 0$

(d) Find the total distance traveled by the particle in the first 6 seconds.

$$s(0) = 0 + 0 + 0 = 0$$

$$s(2) = 2^3 - 12 \cdot 2^2 + 36 \cdot 2 = 32$$

$$s(6) = 0$$

$$\text{Total distance} = |s(2) - s(0)| + |s(6) - s(2)|$$

$$= 32 + 32 = \boxed{64}$$