Math 135, Calculus 1, Fall 2020

11-09: Extreme Values (Section 4.2)

The **derivative** $f'(x)$ of a function $y = f(x)$ gives:

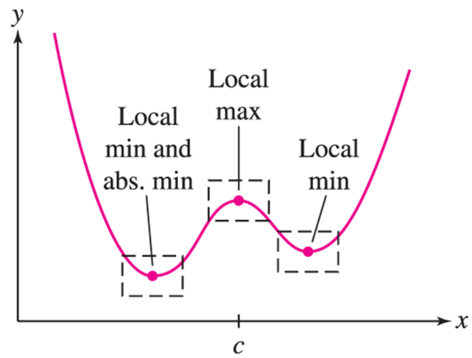
- the slope of the tangent line
- the instantaneous rate of change of y with respect to x

Today, we will begin our discussion of the application of the derivative to **optimization** problems, finding the maximum or minimum values of a function.

A. Local Extrema

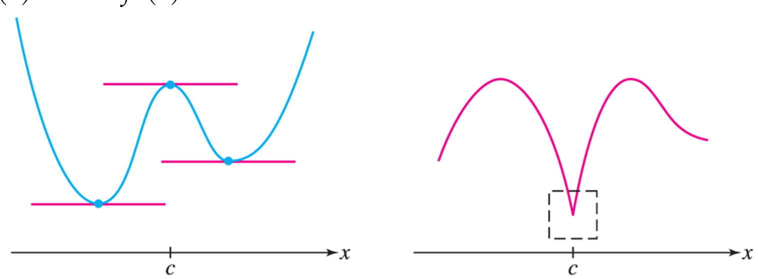
Definition 1. We say that $f(c)$ is a

- **local minimum** occuring at $x = c$ if $f(c) \leq f(x)$ for "all x near c"
- **local maximum** occuring at $x = c$ if $f(c) \geq f(x)$ for "all x near c"



We will spend a good amount of time in the future **finding** and **classifying** these local extrema.

Theorem 2 (Fermat's Theorem on Local Extrema)**.** *If $f(c)$ is a local max or min, then c is a* ***critical point*** *of f: either $f'(c) = 0$ or $f'(c)$ DNE.*



(A) Tangent line is horizontal at the local extrema.

(B) This local minimum occurs at a point where the function is not differentiable

Thus we should think of **critical points** as *potential local extrema*.

Exercise 1. Find the critical points and the associated function values for:

(a) $f(x) = x^2 - 2x + 4$

$f'(x) = 2x - 2 = 0$

$\boxed{x = 1}$

$f(1) = (1)^2 - 2(1) + 4 = \boxed{3}$

(b) $f(x) = x^{-1} - x^{-2}$

$f'(x) = -x^{-2} + 2x^{-3} = 0$

$\left(\frac{-1}{x^2} + \frac{2}{x^3} = 0\right) x^3$

$-x + 2 = 0$

$\boxed{2 = x}$

$x^{-2}\left(-1 + \frac{2}{x}\right) = 0$

~~$x^{-2} = 0$~~

$-1 + \frac{2}{x} = 0$

$\boxed{x = 2}$

$f(2) = \frac{1}{2} - \frac{1}{4} = \boxed{\frac{1}{4}}$

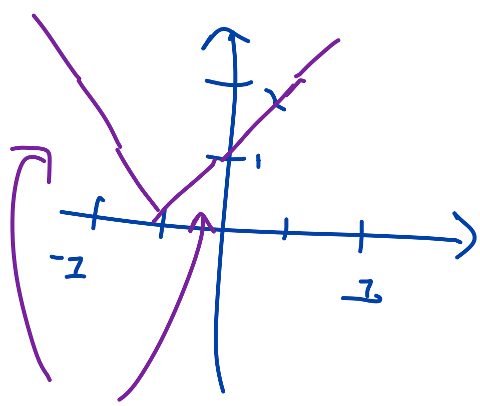
(c) $f(x) = |2x + 1|$

$f(x) = \begin{cases} 2x+1 & 2x+1 \geq 0 \iff x \geq -1/2 \\ -2x-1 & 2x+1 \leq 0 \iff x \leq -1/2 \end{cases}$

$f'(x) = \begin{cases} 2 & x > -1/2 \\ -2 & x < -1/2 \\ \text{DNE} & x = -1/2 \end{cases}$

CP: $\boxed{x = -1/2}$

$f(-1/2) = \boxed{0}$



linear function except when the sub changes: $2x + 1 = 0$

$\boxed{x = -1/2}$

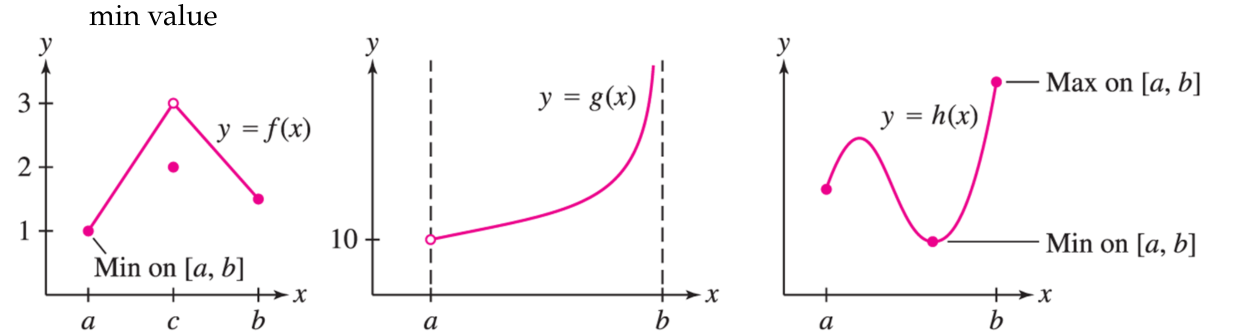
B. Absolute Extrema

Definition 3. Let f be a function defined on an interval I, and let a be in I. We say that $f(a)$ is the

- **absolute minimum** of f on I if $f(a) \le f(x)$ for all x in I
- **absolute maximum** of f on I if $f(a) \ge f(x)$ for all x in I

Example 4. Not every function has an absolute max or min:

- The function $f(x) = x$ on $(-\infty, \infty)$ increases without bound as $x \to \infty$, and descreases without bound as $x \to -\infty$
- If f is **discontinuous** or defined on an **open interval**, it need not achieve a max value or a min value



(A) Discontinuous function with no max on $[a, b]$, and a min at $x = a$.

(B) Continuous function with no min or max on the open interval (a, b).

(C) Every continuous function on a closed interval $[a, b]$ has both a min and a max on $[a, b]$.

Theorem 5 (Extreme Value Theorem on a Closed Interval). *If f is continous on closed interval $I = [a, b]$, that f acheives both an absolute max and an absolute min on $[a, b]$. Moreover, these occur at either critical points or the endpoints a, b.*

Exercise 2. Find the absolute extreme values of $f(x)$ on the interval given by comparing values at the critical points and endpoints:

(a) $f(x) = x^2 - 2x + 4$, $I = [0, 2]$

$f(0) = 4$

$f(2) = 4 - 4 + 4 = 4$

$f(1) = 3$

max : 4

min : 3

(b) $f(x) = x^{-1} - x^{-2}$, $I = [\not{0}\,1, 4]$

$f(1) = \frac{1}{1} - \frac{1}{1} = 0$

$f(4) = \frac{1}{4} - \frac{1}{16} = \frac{3}{16}$

$f(2) = \frac{1}{4}$

max: 1/4

min: 0

(c) $f(x) = |2x + 1|$, $I = [1, 3]$

$f(1) = |2(1) + 1| = 3$

$f(3) = |2 \cdot (3) + 1| = 7$

CP: ~~$x = -1/2$~~ outside interval

max: 7

min: 3

Theorem 6 (Rolle's Theorem)**.** *Suppose f is continuous on $[a, b]$ and differentiable on (a, b). If $f(a) = f(b)$, then there exists a number c between a and b such that $f'(c) = 0$.*

Exercise 3. Verify Rolle's Theorem for $f(x) = \sin(x)$ on $[\pi/4, 3\pi/4]$: check that $f(a) = f(b)$, and find the value c in $(\pi/4, 3\pi/4)$ such that $f'(c) = 0$.

Exercise 4. Use Rolle's Theorem to prove that $f(x) = x^3 + 3x^2 + 6x$ has precisely one real root:

(a) Find points $x = a$ and $x = b$ such that $f(a) < 0$ and $f(b) > 0$.

(b) By the **Intermediate Value Theorem**, there thus exists some point c in (a, b) with $f(c) = 0$, so $f(x)$ has at least one real root. (We do not need to find the exact value of $x = c$.)

(c) By Rolle's Theorem, what would have to be true about f if it had another root at $x = d$?

(d) Why is the above not possible?

Exercise 5. Find the absolute extreme values of $f(x)$ on the interval given by comparing values at the critical points and endpoints:

(a) $f(x) = \dfrac{x^2+1}{x-4}$, $I = [5,6]$.

(b) $f(x) = x + \sin x$, $I = [0, 2\pi]$

(c) $f(x) = \dfrac{\ln x}{x}$, $I = [1,3]$