Math 135, Calculus 1, Fall 2020

11-09: Extreme Values (Section 4.2)

The **derivative** $f'(x)$ of a function $y = f(x)$ gives:

- the slope of the tangent line
- the instantaneous rate of change of y with respect to x

A. Extreme Value Theorem

Theorem 1 (Extreme Value Theorem on a Closed Interval)**.** *If f is continous on closed interval $I = [a, b]$, that f acheives both an absolute max and an absolute min on $[a, b]$. Moreover, these occur at either critical points or the endpoints a, b.*

Exercise 1. Find the absolute extreme values of $f(x)$ on the given interval by comparing values at the critical points and endpoints:

(a) $f(x) = x + \sin x$, $I = [0, 2\pi]$

$f'(x) = 1 + \cos x \stackrel{!}{=} 0$

$\cos x = -1$

$x = \pi, 3\pi, 5\pi \ldots$

CP in $[0, 2\pi]$

$f(0) = 0 + \sin(0) = 0$

$f(2\pi) = 2\pi + \sin(2\pi) = 2\pi$

$f(\pi) = \pi + \sin(\pi) = \pi + 1$

$\boxed{\text{Max: } 2\pi \quad \text{Min: } 0}$

(b) $f(x) = \dfrac{1 - x}{x^2 + 3x}$, $I = [1, 4]$

$$f'(x) = \frac{(-1)(x^2+3x) - (1-x)(2x+3)}{(x^2+3x)^2}$$

DNE: $(x^2+3x)^2 = 0$

$x(x+3) = 0$

$\cancel{\boxed{x=0}}$ $\cancel{\boxed{x=-3}}$

$f'(x) = 0$

$-x^2 - 3x - 2x - 3 + 2x^2 + 3x = 0$

$x^2 - 2x - 3 = 0$

$(x-3)(x+1) = 0$

$\boxed{x = 3}$ $\cancel{\boxed{x = -1}}$

$f(1) = 0$ MAX

$f(4) = \frac{-3}{28}$ MIN

$f(3) = \frac{-2}{18} = -\frac{1}{9}$

(c) $f(x) = x \cdot \ln x$, $I = [1, 3]$

- $f'(x) = (1)\ln(x) + x \cdot \frac{1}{x} = \ln(x) + 1$
- DNE: $x \leq 0$ (same as $f(x)$) ✗ not in interval
- $f'(x) = 0$: $\ln(x) + 1 = 0$

$\ln(x) = -1$

$\cancel{\boxed{x = e^{-1}}}$ not in interval

$f(1) = 1 \cdot \ln(1) = 0$ MIN

$f(3) = 3 \cdot \ln(3) > 0$ MAX

Theorem 2 (Rolle's Theorem). *Suppose f is continuous on $[a, b]$ and differentiable on (a, b). If $f(a) = f(b)$, then there exists a number c between a and b such that $f'(c) = 0$.*

Exercise 2. Verify Rolle's Theorem for $f(x) = \sin(x)$ on $[\pi/4, 3\pi/4]$: check that $f(a) = f(b)$, and find the value c in $(\pi/4, 3\pi/4)$ such that $f'(c) = 0$.

- $f(a) = \sin(\pi/4) = \sqrt{2}/2$
- $f(b) = \sin(3\pi/4) = \sqrt{2}/2$ ✓
- $f'(x) = \cos x \overset{!}{=} 0$

$x = \ldots, \pi/2, 3\pi/2, \ldots$

$\boxed{c = \pi/2}$

Exercise 3. Use Rolle's Theorem to prove that $f(x) = x^3 + 3x^2 + 6x$ has precisely one real root:

(a) Find points $x = a$ and $x = b$ such that $f(a) < 0$ and $f(b) > 0$.

$f(-1) = -1 + 3 - 6 = -4$

$f(1) = 1 + 3 + 6 = 10$

$f(0) = 0$ ✓ ROOT

(b) By the **Intermediate Value Theorem**, there thus exists some point c in (a, b) with $f(c) = 0$, so $f(x)$ has at least one real root. (We do not need to find the exact value of $x = c$.)

(c) By Rolle's Theorem, what would have to be true about f if it had another root at $x = d$?

If $f(d) = 0$, then there must exist some x-value c in $(0, d)$ s.t. $f'(c) = 0$.

(d) Why is the above not possible?

- $f'(x) = 3x^2 + 6x + 6 \overset{!}{=} 0$

$x^2 + 2x + 2 = 0$

$$x = \frac{-2 \pm \sqrt{2^2 - 4(1)(2)}}{2} = \frac{-2 \pm \sqrt{4 - 8}}{2}$$ X not possible!

- $f(x)$ has no x-values when the derivative is 0.