Math 135, Calculus 1, Fall 2020

11-13: First Derivative Test (Section 4.3)

The **derivative** $f'(x)$ of a function $y = f(x)$ gives:

- the slope of the tangent line
- the instantaneous rate of change of y with respect to x

The goal of today's class is understand how we can use the first derivative to get information about the original function.

Important result:

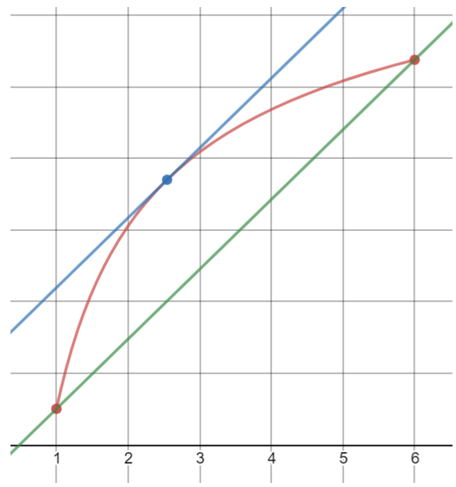
Theorem 1 (Mean Value Theorem (MVT))**.** *If a function f is continuous on the closed interval $[a, b]$ and differentiable on (a, b), then there exists an x-value $c \in (a, b)$ such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

i.e.

- *the instantaneous rate of change / slope of the tangent line at $x = c$, and*
- *the average rate of change / slope of the secant line over the interval $[a, b]$*

*are **equal**.*



Using the MVT, we can show the first derivative indicates whether the function is increasing, decreasing, or neither:

$$f'(x) > 0 \text{ for } x \in (a, b) \Rightarrow f \text{ is } \textbf{increasing} \text{ on } (a, b)$$
$$f'(x) < 0 \text{ for } x \in (a, b) \Rightarrow f \text{ is } \textbf{decreasing} \text{ on } (a, b)$$
$$f'(c) = 0 \Rightarrow c \text{ is a } \textbf{critical point} \text{ of } f$$

We can use this to **classify** when a critical point is a **local max** or **local min**:

First Derivative Test. Suppose that $x = c$ is a critical point of f.

$$f'(x) \text{ changes from } + \text{ to } - \text{ at } c \quad \Rightarrow \quad c \text{ is a } \textbf{local max}$$
$$f'(x) \text{ changes from } - \text{ to } + \text{ at } c \quad \Rightarrow \quad c \text{ is a } \textbf{local min}$$
$$f'(x) \text{ does not change sign at } c \quad \Rightarrow \quad c \text{ is } \textbf{not a local extremum}$$

Example 2. Let $f(x) = x^3 - 3x^2 - 45x + 5$. Together, let's find the critical points of f, and classify them using the First Derivative Test. On what interval(s) is f increasing? decreasing? Use this information to sketch a graph of f.

$$f'(x) = 3x^2 - 6x - 45 \overset{!}{=} 0$$
$$x^2 - 2x - 15 = 0$$
$$(x-5)(x+3) = 0$$

CP: $\boxed{x=5}$ $\boxed{x=-3}$

$f'(x) = (x-5)(x+3)$

$f'(-4) = (-)(-) = +$

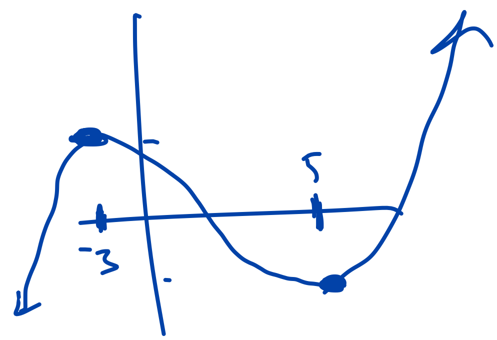
$f'(0) = (-)(+) = -$

$f'(6) = (+)(+) = +$

$-4 \quad -3 \quad 0 \quad 5 \quad 6$

$f'(x)$: $+ \quad 0 \quad - \quad 0 \quad +$

$\boxed{x=-3 \text{ local max}}$ $\boxed{x=5 \text{ local min}}$



Exercise 1. For each of the following functions:

- Find the critical points of $f(x)$
- Find the intervals on which $f(x)$ is increasing or decreasing.
- Classify the critical points using the First Derivative Test

(a) $f(x) = \dfrac{x^2 - 8x}{x+1}$

$$f'(x) = \frac{(2x-8)(x+1) - (x^2-8x)}{(x+1)^2}$$
$$= \frac{2x^2 + 2x - 8x - 8 - x^2 + 8x}{(x+1)^2}$$
$$= \frac{x^2 + 2x - 8}{(x+1)^2} = \frac{(x+4)(x-2)}{(x+1)^2}$$

DNE: $x = -1$

$f'(x) = 0$: $x = 2$, $x = -4$

$-4 \quad -1 \quad 2$

$f'(x)$: $/ \ 0 \ \backslash \quad 0 \quad \backslash \ 0$

(b) $f(x) = (x^2 - 2x)e^x$

$f'(x) = e^x(x^2 - 2) \overset{!}{=} 0$

$\cancel{e^x = 0}$ $\quad x^2 - 2 = 0$

(c) $f(x) = 15x^3 - x^5$