# Math 135, Calculus 1, Fall 2020

11-13: First Derivative Test (Section 4.3)

The **derivative** $f'(x)$ of a function $y = f(x)$ gives:

- the slope of the tangent line
- the instantaneous rate of change of $y$ with respect to $x$

The goal of today's class is understand how we can use the first derivative to get information about the original function.

Important result:

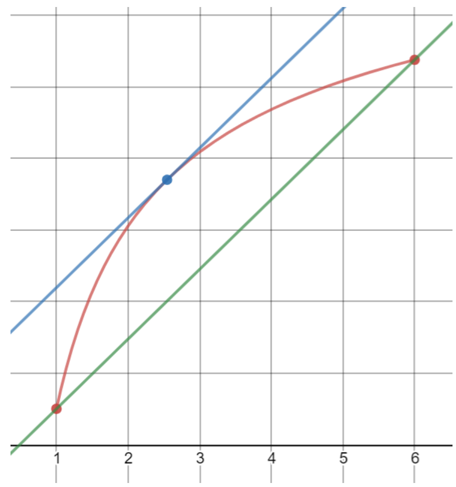
**Theorem 1** (Mean Value Theorem (MVT))**.** *If a function $f$ is continuous on the closed interval $[a, b]$ and differentiable on $(a, b)$, then there exists an $x$-value $c \in (a, b)$ such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

*i.e.*

- *the instantaneous rate of change / slope of the tangent line at $x = c$, and*
- *the average rate of change / slope of the secant line over the interval $[a, b]$*

*are **equal**.*




Using the MVT, we can show the first derivative indicates whether the function is increasing, decreasing, or neither:

$$f'(x) > 0 \text{ for } x \in (a, b) \Rightarrow f \text{ is } \textbf{increasing} \text{ on } (a, b)$$
$$f'(x) < 0 \text{ for } x \in (a, b) \Rightarrow f \text{ is } \textbf{decreasing} \text{ on } (a, b)$$
$$f'(c) = 0 \Rightarrow c \text{ is a } \textbf{critical point} \text{ of } f$$

We can use this to **classify** when a critical point is a **local max** or **local min**:

**First Derivative Test.** Suppose that $x = c$ is a critical point of $f$.

$$f'(x) \text{ changes from } + \text{ to } - \text{ at } c \quad \Rightarrow \quad c \text{ is a } \textbf{local max}$$
$$f'(x) \text{ changes from } - \text{ to } + \text{ at } c \quad \Rightarrow \quad c \text{ is a } \textbf{local min}$$
$$f'(x) \text{ does not change sign at } c \quad \Rightarrow \quad c \text{ is } \textbf{not a local extremum}$$

**Example 2.** Let $f(x) = x^3 - 3x^2 - 45x + 5$. Together, let's find the critical points of $f$, and classify them using the First Derivative Test. On what interval(s) is $f$ increasing? decreasing? Use this information to sketch a graph of $f$.

CP: $f'(x) = 3x^2 - 6x - 45 \stackrel{!}{=} 0$

$x^2 - 2x - 15 = 0$

$(x-5)(x+3) = 0$

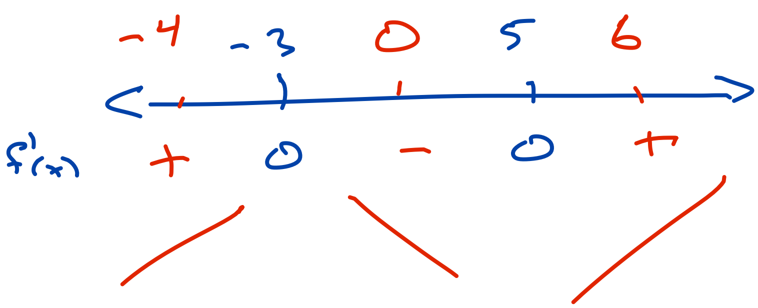
$\boxed{x=5}$ $\boxed{x=-3}$

$f'(x) = (x-5)(x+3)$

$f'(-4) = (-)(-) > 0$

$f'(0) = (-)(+) < 0$

$f'(6) = (+)(+) > 0$




$\boxed{x = -3 \text{ local max}}$

$\boxed{x = 5 \text{ local min}}$

**Exercise 1.** For each of the following functions:

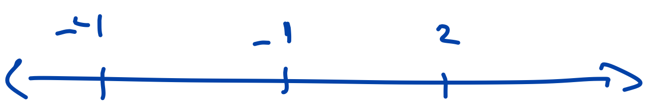
- Find the critical points of $f(x)$
- Find the intervals on which $f(x)$ is increasing or decreasing.
- Classify the critical points using the First Derivative Test

(a) $f(x) = \dfrac{x^2 - 8x}{x + 1}$

$$f'(x) = \frac{(2x-8)(x+1) - x^2 + 8x}{(x+1)^2}$$

$$= \frac{2x^2 + 2x - 8x - 8 - x^2 + 8x}{(x+1)^2}$$

$$= \frac{x^2 + 2x - 8}{(x+1)^2} = \frac{(x+4)(x-2)}{(x+1)^2}$$



$0 \longleftarrow \qquad \longrightarrow \infty$

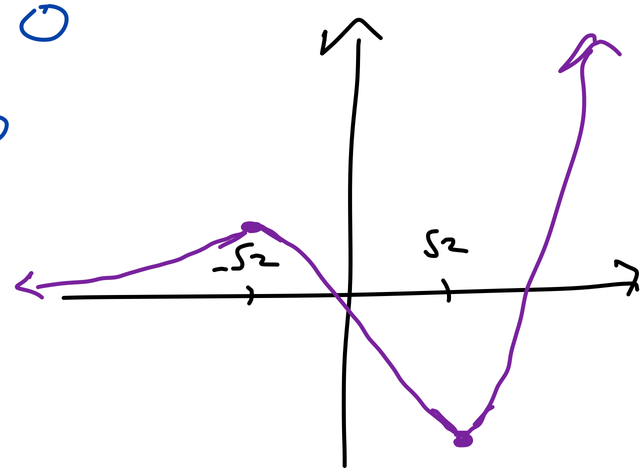
(b) $f(x) = (x^2 - 2x)e^x$

$$f'(x) = (2x-2)e^x + (x^2-2x)e^x$$
$$= (x^2-2)e^x \stackrel{!}{=} 0$$

$x^2 - 2 = 0$ $\qquad$ $\cancel{e^x = 0}$ never

$x^2 = 2$

$x = \pm\sqrt{2}$




| | $-3$ | $-\sqrt{2}$ | $0$ | $\sqrt{2}$ | $3$ |
|---|---|---|---|---|---|
| $f'(x)$ | $+$ | $0$ | $-$ | $0$ | $+$ |

$f(-\sqrt{2}) = (2+2\sqrt{2})e^{-\sqrt{2}}$

$f(\sqrt{2}) = (2-2\sqrt{2})e^{\sqrt{2}}$

(c) $f(x) = 15x^3 - x^5$

$f'(x) = 45x^2 - 5x^4 = 0$

$5x^2 = 0$ $\qquad$ $9 - x^2 = 0$

$x = 0$ $\qquad$ $x = \pm 3$

| | | $-3$ | | $0$ | | $3$ | |
|---|---|---|---|---|---|---|---|
| $f'(x)$ | $-$ | $0$ | $+$ | $0$ | $+$ | $0$ | $-$ |

$x = -3$ local min

$x = 3$ local max

$x = 0$ neither