Math 135, Calculus 1, Fall 2020

11-30: L'Hôpital's Rule

The **derivative** $f'(x)$ of a function $y = f(x)$ gives:

- the slope of the tangent line
- the instantaneous rate of change of y with respect to x

Common Problem. Suppose we want to compute the limit of a rational function

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

but when we plug in $x = a$ (including $a = \pm\infty$), we get one of the following **indeterminant forms**:

$$\frac{0}{0} \quad \text{or} \quad \frac{\pm\infty}{\infty}$$

In this case, we can apply L'Hôpital's Rule to help compute this limit.

Theorem 1 (L'Hôpital's Rule, Guillaume François Antoine Marquis de L'Hôpital, 1696)**.** *If f and g are differentiable functions such that $\frac{f(a)}{g(a)}$ is one of the above indeterminant forms, then*

$$\boxed{\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}.}$$

- You may have to apply the rule multiple times to determine the limit
- The rule was actually discovered by Bernoulli in 1964.

Example 2. Consider $\lim_{x \to 0} \frac{\sin x}{x}$, where x is in radians. Plugging in $x = 0$ yields the indeterminant form $\frac{0}{0}$, so L'Hôpital's Rule applies. Thus we have

$$\lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} \frac{\cos x}{1} = \frac{\cos 0}{1} = 1.$$

This matches our calculation from Section 2.6.

Exercise 1. Use L'Hôpital's Rule to compute the other important trig limit

$$\lim_{x \to 0} \frac{1 - \cos x}{x} = \frac{1 - \cos 0}{0} = \frac{1-1}{0} = \frac{0}{0} \quad X$$

$$\Rightarrow \lim_{x \to 0} \frac{1 - \cos x}{x} \overset{L'H}{=} \lim_{x \to 0} \frac{0 + \sin x}{1} = \sin 0 = \boxed{0} \checkmark$$

Exercise 2. Ccompute the following limits. Make sure to **first check** that the rule actually applies.

(a) $\lim_{x\to 2} \frac{x^3-8}{x^4-x^3-8}$ $\overset{L'H}{=} \lim_{x\to 2} \frac{3x^2}{4x^3-3x^2} = \frac{12}{32-12} = \frac{12}{20} = \boxed{\frac{3}{5}}$

$= \frac{8-8}{16-8-8} = \frac{0}{0}$ ✓

(b) $\lim_{x\to 0} \frac{1-\cos x}{x^2}$ $\overset{L'H}{=} \lim_{x\to 0} \frac{\sin x}{2x} \overset{L'H}{=} \lim_{x\to 0} \frac{\cos x}{2} = \frac{\cos(0)}{2} = \boxed{\frac{1}{2}}$

$= \frac{1-1}{0} = \frac{0}{0}$ ✓ $\qquad = \frac{0}{2\cdot 0} = \frac{0}{0}$ ✓

(c) $\lim_{x\to 3} \frac{x-3}{x}$ $= \frac{3-3}{3} = \frac{0}{3} = \boxed{0}$ No L'H !

(d) $\lim_{x\to\infty} \frac{\ln x}{\sqrt{x}}$ $= \lim_{x\to\infty} \frac{1/x}{1/2\sqrt{x}} = \lim_{x\to\infty} \frac{1}{x} \cdot \frac{2\sqrt{x}}{1} = \lim_{x\to\infty} \frac{2\sqrt{x}}{x}$

$= \lim_{x\to\infty} \frac{2}{\sqrt{x}} = \frac{2}{\infty} = \boxed{0}$

(e) $\lim_{x\to\infty} \frac{e^x}{x}$ $\overset{L'H}{=} \lim_{x\to\infty} \frac{e^x}{1} = e^\infty = \boxed{\infty}$

$= \frac{e^\infty}{\infty} = \frac{\infty}{\infty}$ ✓

Exercise 3. Use L'Hôpital's Rule to find any horizontal asymptotes of the following functions.

(a) $f(x) = \frac{8x^3 - 4x + \pi}{-3x^3 + 7x^2 + 5}$

$\lim_{x\to\pm\infty} f(x) \overset{L'H}{=} \lim_{x\to\pm\infty} \frac{24x^2-4}{-9x^2+14x} \overset{L'H}{=} \lim_{x\to\pm\infty} \frac{48x}{-18x+14} \overset{L'H}{=} \lim_{x\to\pm\infty} \frac{48}{-18} = \boxed{\frac{-8}{3}}$

(b) $g(x) = \frac{3e^{2x} + 5x}{e^{2x} + 8x}$

$\lim_{x\to\pm\infty} g(x) \overset{L'H}{=} \lim_{x\to\pm\infty} \frac{6e^{2x}+5}{2e^{2x}+8} \overset{L'H}{=} \lim_{x\to\pm\infty} \frac{12e^{2x}}{4e^{2x}}$

$= \lim_{x\to\pm\infty} \frac{12}{4} = \boxed{3}$